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An Improved Method of Numerical Ray Tracing through Electron Lenses

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ABSTRACT. An improved method is described for the tracing of electron trajectories through electron lenses by integrating the general ray equation with the help of Taylor's series, taking the series up to and including fourth-order terms. The paraxial trajectory is obtained by progressing from point P_n on the trajectory to point P_{n+1} with the help of recurrence formulae $r_{n+1} = Q_1 r_n + Q_2 r'_n$ and $r'_{n+1} = Q_3 r'_n + Q_4 r''_n$, where the coefficients $Q_1, \dots, Q_4$ are functions of the electrostatic or magnetic field distribution along the axis. The deviation of the first-order trajectory from the paraxial trajectory is built up with the help of similar recurrence formulae. The method is first developed for combined electrostatic and magnetic lenses, using the "equivalent potential" $U = \Phi - eA^2/2m$, and then specialized for pure magnetic or pure electrostatic lenses. The case of space charge within lenses is also considered. Comparison of the improved method with other methods shows that it is faster and capable of high accuracy. The factors influencing the accuracy are considered in detail and a short-cut method using extrapolation is given. A fully calculated example is appended.

§1. INTRODUCTION

AN axially symmetrical electrostatic or magnetic field has the property of a lens for an electron beam passing through it. The general behaviour of such electron lenses is well known and is set out in several textbooks on electron optics. Information on the *detailed* properties of *specific* electron lenses, i.e. lenses of certain prescribed geometric patterns, dimensions and field distributions, is however still very incomplete. Only a few types have been studied experimentally, and whilst several papers on computed lens properties have been published, only three of these, by Glaser (1941) on magnetic lenses of the field form $H = H_0/\{1 + (z/a)^2\}$, by Ramberg (1942) on the axial aberrations of four idealized lenses, and by Rüdénberg (1948) on a special form of unipotential lens, are comprehensive. In view of the importance of more detailed knowledge of the properties of strong electron lenses, including their aberrations, for the further development of applied electron optics, a programme of investigation was commenced some time ago in this laboratory.

It was considered essential to study theoretically the properties of *real lenses*, without the introduction of far-reaching idealizations at the very beginning, and without the limitations which such idealizations would impose. The differential

equations which describe the path of an electron through an electron lens cannot then be solved analytically, and a numerical solution for the electron trajectory has to be sought for the given field distribution and the given initial conditions. From a number of electron trajectories other information about the lens, such as paraxial focal length, position and shape of principal surfaces, spherical and chromatic aberration constants, etc., can then be derived. The use of the trajectory method (as opposed to certain integral formulae) to derive the aberrations may permit a more detailed insight into the connection between aberrations and field shape, as is desirable for the progress of electron lens design.

Methods of computing the electron trajectory have been given previously, but only a few authors deal with methods suitable for non-paraxial rays. The most notable of these was given by Goddard (1944) and applied by Goddard and Klemperer (1944); it is suitable for ray bundles of very large angular aperture. The method of Motz and Klanfer (1946) can be applied to the same end, but proves more lengthy. The usefulness of the interesting and straightforward method of Cosslett (1946, p. 97) is restricted to the paraxial case, as the formulae given for its extension to the extra-paraxial region are incomplete.

A numerical solution has to be valued according to its accuracy and to the effort required to obtain the solution. The ray-tracing method forming the subject of this paper will be found an improvement in these respects.

In the following, it will be assumed that the field distribution within the lens as a function of r and z , or the field along the axis and its derivatives, are known with sufficient accuracy. The field distribution can be obtained from the lens geometry and the lens potentials, or the excitation in magnetic lenses, either by direct measurements or by using analogue or relaxation methods.

The electromagnetic system of units (E.M.U.) is used in this paper unless stated otherwise.

§2. DERIVATION OF THE IMPROVED RAY-TRACING METHOD

This ray-tracing method is a further development of the earlier applications of Taylor's series to the solution of the paraxial ray equation by Maloff and Epstein (1934, 1938) and by Cosslett (1946, p. 37). The approach is different in starting from the more general equations, applying anywhere within the field and specializing at a later stage towards the paraxial case. Combined electrostatic and magnetic fields will be considered initially; their effect will be separated only in the later applications of the method. The trajectory will be obtained with the help of recurrence formulae, which can also be applied to obtain the deviation of a "first-order trajectory" from a paraxial trajectory.

The general equation of motion of an electron in an electromagnetic field is

$$\mathbf{F} = m\mathbf{a} = -e[\mathbf{v} \times \mathbf{H}] + e \text{grad } \Phi, \quad \dots\dots (1)$$

where $\mathbf{F}$ is the force acting on the electron, m its mass, $\mathbf{a}$ its acceleration, $-e$ its charge, $\mathbf{v}$ its velocity, $\mathbf{H}$ the magnetic field strength and Φ the electrostatic potential.

The magnetic field is replaced by the vector potential $\mathbf{A}$, defined by $\mathbf{H} = \text{curl } \mathbf{A}$, making use of the condition of rotational symmetry

$$\frac{\partial H}{\partial \phi} = 0, \quad \frac{\partial \Phi}{\partial \phi} = 0, \quad \mathbf{A} = A_\phi = A.$$

It will further be assumed that the rays considered are contained in a meridional plane, i.e. a plane passing through the optical axis; it can be shown that rays which

are in a meridional plane at any time will always remain in a meridional plane, although this plane will, in a magnetic field, rotate about the axis by an angle ψ as the electron moves through the field. The exclusion of skew rays simplifies the treatment considerably, and does not lead to any loss of information about the lens under investigation, up to and including first-order aberrations (see § 7).

The equations of motion can then be written, using the cylindrical coordinates z, r, ψ ,

$$\ddot{z} = \left(\frac{e}{m}\right) \frac{\partial U}{\partial z} \quad \dots\dots (2a)$$

$$\ddot{r} = \left(\frac{e}{m}\right) \frac{\partial U}{\partial r} \quad \dots\dots (2b)$$

$$\dot{\psi} = \left(\frac{e}{m}\right) \frac{A}{r} \quad \dots\dots (2c)$$

$$\text{where} \quad U = \Phi - eA^2/2m \quad \dots\dots (3)$$

is the "equivalent potential" (Brüche and Scherzer 1934, p. 116).

Eliminating the time from equation (2) with the help of the energy equation $\frac{1}{2}m(\dot{z}^2 + \dot{r}^2 + r^2\dot{\psi}^2) = e\Phi$ and the relation $\dot{r} = r''\dot{z}^2 + r'\dot{z}$, the general ray equation is obtained:

$$r'' - \frac{1+r'^2}{2U} \left\{ \frac{\partial U}{\partial r} - r' \frac{\partial U}{\partial z} \right\} = 0. \quad \dots\dots (4)$$

This general ray equation determines the electron trajectory in the plane rotating about the axis by an angle ψ , the angle of rotation being given by a first-order differential equation, derived from equation (2c) by a similar process as the ray equation:

$$\psi' = \left(\frac{e}{2m}\right)^{\frac{1}{2}} \frac{A}{r} (1+r'^2)^{\frac{1}{2}} U^{-\frac{1}{2}}. \quad \dots\dots (5)$$

The general ray equation (4) is the basis of the further considerations of this paper. It may be noted that the only two basic assumptions in the derivation of equation (4) are: (i) there are no conductors *within* the space whose field is considered; (ii) axial symmetry. (Equation (4) is valid also for skew rays, provided a term expressing the angular momentum outside the lens field is included in the expression for U , equation (3).)

The general ray equation (4) will now be integrated step by step by application of the Taylor series, proceeding from point P_n , coordinates z_n, r_n, r'_n , etc., to point P_{n+1} , coordinates $z_{n+1}, r_{n+1}, r'_{n+1}$, etc. The field terms $U, r^{-1} \partial U / \partial r$ and $\partial U / \partial z$ are functions of z and r ;* we shall replace the continuous curves representing these functions by stepped lines, each step of constant field value coinciding with a z interval. This will lead to a slight error in the curvature of the trajectory, determined by r'' , at the beginning of the interval, which is cancelled (except for higher order terms) by an error of opposite sign near the end of the interval, if the constant field values assigned to the $(n+1)$ th interval are the mean of the field values at P_n and P_{n+1} . It is found most convenient in practice

* Near the axis, these terms are functions of z only, since for $r \ll 1$, $U \simeq \Phi_0$,

$r^{-1} \partial U / \partial r \simeq \frac{1}{2}(\Phi_0'' + eH_0^2/2m)$, and $\partial U / \partial z \simeq \Phi_0'$.

to choose the geometric mean values*. Writing U_r for $\partial U/\partial r$ and U_z for $\partial U/\partial z$, etc., the following table can be written down for the values needed in Taylor's series:

r, r' = initial values,

$$r'' = \frac{1}{2U} \left\{ \left[r \left(\frac{U_r}{r} \right) - r' U_z \right] + \left[r'^2 \left(\frac{U_r}{r} \right) - r'^3 U_z \right] \right\},$$

$$r''' = \frac{1}{(2U)^2} \left\{ -r \left(\frac{U_r}{r} \right) U_z + r' \left[2U \left(\frac{U_r}{r} \right) + U_z^2 \right] \right\}$$

+ (third and higher order terms),

$$r'''' = \frac{1}{(2U)^3} \left\{ r \left[2U \left(\frac{U_r}{r} \right)^2 + \left(\frac{U_r}{r} \right) U_z^2 \right] - r' \left[4U \left(\frac{U_r}{r} \right) U_z + U_z^3 \right] \right\}$$

+ (third and higher order terms).

The first four terms of the Taylor series for r and r' are

$$r_{n+1} = r_n + r'_n \Delta z + r''_n \frac{\Delta z^2}{2} + r'''_n \frac{\Delta z^3}{6} + \dots,$$

$$r'_{n+1} = r'_n + r''_n \Delta z + r'''_n \frac{\Delta z^2}{2} + r''''_n \frac{\Delta z^3}{6} + \dots$$

If all terms but that of first order are neglected in r'''' and r''''' , these two series developments for r_{n+1} and r'_{n+1} contain all terms up to and including the fourth order, assuming Δz to be of the same order as r and r' . It was found that this gives good accuracy with reasonably large intervals, of the order of $\Delta z = \frac{1}{10}(\frac{1}{2}D)$, where D is the maximum physical diameter of the lens. For still larger z intervals, the third order term would have to be retained in r''' , and the first-order term of r'''' would have to be included in the series for r'_{n+1} if an accuracy to within 1% in the final steps of the traced trajectory is required, but the complexity of the expression grows fast with increasing order of terms. Obviously, if the z intervals were taken much smaller, the series could be broken off earlier, but this again involves more computational work.

Introducing the above expressions for r, r', r'', r''' and r'''' into the Taylor series and re-arranging the terms, r_{n+1} and r'_{n+1} are obtained as functions of r_n and r'_n :

$$\begin{aligned} r_{n+1} = & r_n \left[1 + \frac{\Delta z^2}{4U} \left(\frac{U_r}{r} \right) - \frac{\Delta z^3}{24U^2} \left(\frac{U_r}{r} \right) U_z \right] \\ & + r'_n \Delta z \left[1 - \frac{U_z}{4U} \Delta z + \frac{1}{12U} \left(\frac{U_r}{r} \right) \Delta z^2 + \frac{U_z^2}{24U^2} \Delta z^2 \right] \\ & + \frac{r_n'^2}{2U} \left[r_n \left(\frac{U_r}{r} \right) - r'_n U_z \right] \frac{\Delta z^2}{2} + \dots \end{aligned}$$

$$\begin{aligned} \text{and } r'_{n+1} = & r'_n \left[1 - \frac{\Delta z}{2U} U_z + \frac{\Delta z^2}{4U} \left(\frac{U_r}{r} \right) + \frac{\Delta z^2 U_z^2}{8U^2} \right] \\ & + r_n \frac{\Delta z}{2U} \left[\left(\frac{U_r}{r} \right) - \frac{\Delta z}{4U} \left(\frac{U_r}{r} \right) U_z \right] + r'^2 \frac{\Delta z}{2U} \left[r \left(\frac{U_r}{r} \right) - r' U_z \right] + \dots \end{aligned}$$

* Most fields representing electron lenses show an essentially exponential variation in the axial direction, at least in parts of the lens where the field changes rapidly. In this case the relation $\{f(z_1)f(z_2)\}^{\frac{1}{2}} = f((z_1+z_2)/2)$ holds, i.e. the geometric mean gives the value at the mid-interval point exactly, whereas it is still as good an approximation as the arithmetic mean to the "central" value of the field if this should be represented by another type of function. Moreover, in magnetic lenses, we shall require $H^2(z)$, and its geometric mean $H(z_1)H(z_2)$ is much simpler to calculate than $\frac{1}{2}(H^2(z_1) + H^2(z_2))$.

The field values U , U_r/r and U_z are now written more explicitly, using the well-known series expansions for $\Phi(r, z)$ and $A(r, z)$ in terms of the potential $\Phi_0(z)$ and magnetic field strength $H_0(z)$ on the axis, and their derivatives with respect to z (Brüche and Scherzer 1934b, pp. 66, 80):

$$\Phi(r, z) = \Phi_0(z) - \frac{r^2}{4} \Phi_0''(z) + \frac{r^4}{64} \Phi_0''''(z) \mp \dots$$

$$A(r, z) = \frac{r}{2} H_0(z) - \frac{r^3}{16} H_0''(z) \pm \dots$$

Hence, leaving out now the suffixes of Φ_0 and H_0 ,

$$U = \Phi - \frac{r^2}{4} \left[\Phi'' + \frac{e}{2m} H^2 \right] + \frac{r^4}{16} \left[\frac{\Phi''''}{4} + \frac{e}{2m} H H'' \right] + \dots,$$

$$\frac{1}{U} = \frac{1}{\Phi} \left\{ 1 + \frac{r^2}{4} \left[\frac{\Phi''}{\Phi} + \frac{e}{2m\Phi} H^2 \right] + \frac{r^4}{16} \left[\left(\frac{\Phi''}{\Phi} + \frac{e}{2m\Phi} H^2 \right)^2 - \left(\frac{\Phi''''}{4\Phi} + \frac{e}{2m\Phi} H H'' \right) \right] + \dots \right\},$$

$$\left(\frac{U_r}{r} \right) = -\frac{\Phi}{2} \left\{ \left[\frac{\Phi''}{\Phi} + \frac{e}{2m\Phi} H^2 \right] - \frac{r^2}{2} \left[\frac{\Phi''''}{4\Phi} + \frac{e}{2m\Phi} H H'' \right] + \dots \right\},$$

$$U_z = \Phi \left\{ \frac{\Phi'}{\Phi} - \frac{r^2}{4} \left[\frac{\Phi'''}{\Phi} + \frac{e}{m\Phi} H H' \right] + \frac{r^4}{16} \left[\frac{\Phi'''''}{4\Phi} + \frac{e}{2m\Phi} H' H'' + \frac{e}{2m\Phi} H H'' \right] + \dots \right\}$$

These expansions are introduced into the previous expressions for r_{n+1} and r'_{n+1} , but writing now R_{n+1} and R'_{n+1} instead for the general trajectory, reserving the symbols r and r' for the paraxial case. Breaking off after the fourth-order term in Δz , R and R' , these recursion formulae are obtained:

$$R_{n+1} = Q_1 R_n + Q_2 R'_n, \quad \dots \dots (6a)$$

$$R'_{n+1} = Q_3 R'_n + Q_4 R_n + Q_5, \quad \dots \dots (6b)$$

where the coefficients $Q_1 \dots Q_5$ are functions only of the field values. The coefficient Q_5 depends on the field distribution and on the values R_n and R'_n of the trajectory.

The five coefficients are given by the expressions

$$Q_1 = 1 - \frac{1}{8} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \Delta z^2 + \frac{1}{48} \left(\frac{\Phi'}{\Phi} \right) \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \Delta z^3, \quad \dots \dots (7a)$$

$$Q_2 = \Delta z - \frac{1}{4} \left(\frac{\Phi'}{\Phi} \right) \Delta z^2 + \frac{1}{24} \left[\left(\frac{\Phi'}{\Phi} \right)^2 - \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \right] \Delta z^3, \quad \dots \dots (7b)$$

$$Q_3 = 1 - \frac{1}{2} \left(\frac{\Phi'}{\Phi} \right) \Delta z + \frac{1}{8} \left[\left(\frac{\Phi'}{\Phi} \right)^2 - \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \right] \Delta z^2, \quad \dots \dots (7c)$$

$$Q_4 = -\frac{1}{4} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \Delta z + \frac{1}{16} \left(\frac{\Phi'}{\Phi} \right) \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \Delta z^2, \quad \dots \dots (7d)$$

$$Q_5 = \left\{ \left[\frac{1}{8} \left(\frac{\Phi''''}{4\Phi} + \frac{eHH''}{2m\Phi} \right) - \frac{1}{16} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right)^2 \right] R_n^3 \right. \\ \left. + \frac{1}{8} \left[\left(\frac{\Phi'''}{\Phi} + \frac{eHH'}{m\Phi} \right) - \left(\frac{\Phi'}{\Phi} \right) \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \right] R_n^2 R_n' \right. \\ \left. - \frac{1}{4} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) R_n R_n'^2 - \frac{1}{2} \left(\frac{\Phi'}{\Phi} \right) R_n'^3 \right\} \Delta z. \quad \dots\dots (7e)$$

The sets of equations (6) and (7) allow one to trace any ray (excluding skew rays) through any space-charge free electron lens system with good accuracy and a minimum of effort, provided the ray passes through lens regions in which the secondary and higher order aberrations are of little influence. From an inspection of field distributions as well as from experimental evidence (Liebmann 1949) it can be inferred that this region extends to almost half-way between the axis and the lens boundaries, i.e. to a distance $0.2D$ from the axis.

If it is not desired to determine an electron trajectory passing at some distance from the axis for its own sake, but rather to find the deviation of this trajectory from a Gaussian or paraxial trajectory, as is needed in a calculation of the (primary) lens aberration constants, the following procedure gives the best accuracy:

* The paraxial ray r, r' is found by application of formulae (6), with $Q_5 \equiv 0$, as it is this coefficient which expresses the first-order aberrations:

$$r_{n+1} = Q_1 r_n + Q_2 r_n', \quad \dots\dots (8a)$$

$$r_{n+1}' = Q_3 r_n' + Q_4 r_n, \quad \dots\dots (8b)$$

The deviations of the true trajectory from the paraxial one, $\Delta r_n = R_n - r_n$ and $\Delta r_n' = R_n' - r_n'$, are now built up along the paraxial trajectory

$$\Delta r_{n+1} = R_{n+1} - r_{n+1} = Q_1(R_n - r_n) + Q_2(R_n' - r_n')$$

and similarly for $\Delta r_{n+1}'$. Hence

$$\Delta r_{n+1} = Q_1 \Delta r_n + Q_2 \Delta r_n', \quad \dots\dots (9a)$$

$$\Delta r_{n+1}' = Q_3 \Delta r_n' + Q_4 \Delta r_n + Q_5. \quad \dots\dots (9b)$$

As the term Q_5 is a correction term, although a very important one, and as first-order trajectories are mostly considered which do not deviate greatly from the paraxial trajectory, the values of R and R' in the expression for Q_5 can be replaced by the paraxial values r and r' :

$$Q_5 = \left\{ \left[\frac{1}{8} \left(\frac{\Phi''''}{4\Phi} + \frac{eHH''}{2m\Phi} \right) - \frac{1}{16} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right)^2 \right] r_{n+\frac{1}{2}}^3 \right. \\ \left. + \frac{1}{8} \left[\left(\frac{\Phi'''}{\Phi} + \frac{eHH'}{m\Phi} \right) - \left(\frac{\Phi'}{\Phi} \right) \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) \right] r_{n+\frac{1}{2}}^2 r_{n+\frac{1}{2}}' \right. \\ \left. - \frac{1}{4} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} \right) r_{n+\frac{1}{2}} r_{n+\frac{1}{2}}'^2 - \frac{1}{2} \left(\frac{\Phi'}{\Phi} \right) r_{n+\frac{1}{2}}'^3 \right\} \Delta z. \quad \dots\dots (10)$$

The coefficients $Q_1 \dots Q_4$ used in equations (9a) and (9b) are, of course, the same as those used in the determination of the paraxial trajectory. The building-up of the difference of paraxial and first-order trajectories leads to a more precise numerical result for the aberrations than the comparison of the first-order trajectory with a paraxial one, as the aberrations in the image plane are here not found from

a small difference between two much larger terms affected by computational errors. Numerical work has shown that it is possible to evaluate, for example, spherical aberration constants, with an accuracy of within 1% even if the first-order trajectory deviates only very little from the paraxial trajectory, an accuracy which has not been reached in the other published ray-tracing methods.

§ 3. RAY TRACING THROUGH LENSES CONTAINING SPACE CHARGE

It may be useful for some applications to enlarge the scope of the earlier formulae by the inclusion of space charge.

The space-charge density $\rho = \rho(r, z)$ may be written as a power series in r^2 :

$$\rho(r, z) = \rho_0(z) + \rho_1(z)r^2 + \rho_2(z)r^4 + \dots,$$

where the coefficients $\rho_0, \rho_1, \rho_2, \dots$ are functions only of z . The field distribution is then no longer determined by Laplace's equation, but by Poisson's equation $\nabla^2\Phi = -4\pi\rho$.

The potential distribution satisfying this equation may be written as a power series in r^2 , as could be done with the electrostatic potential satisfying Laplace's equation, i.e.

$$\Phi = \Phi_s(z) + a_1(z)r^2 + a_2(z)r^4 + \dots,$$

$\Phi_s(z)$ being the potential on the axis.

Introducing this series into Poisson's equation, and equating the coefficients of equal powers of r^2 , one obtains a series of equations for the coefficients $a_1(z), a_2(z), \dots$. Taking terms up to r^4 , this results in

$$\Phi_s(r, z) = \Phi_s - \left(\frac{\Phi_s''}{4} + \pi\rho_0 \right) r^2 + \left(\frac{\Phi_s''''}{64} + \frac{\pi}{16} \rho_0'' - \frac{\pi}{4} \rho_1 \right) r^4.$$

The value of Φ_s in the presence of space charge is, of course, different from the corresponding value of Φ_0 without space charge, used in the earlier equations. Whilst the recursion formulae (5), (7) and (8) remain unchanged, the new potential $\Phi_s(r, z)$ must be introduced into the coefficients $Q_1 \dots Q_5$. In the presence of space charge, the expression for Q_1 becomes, therefore,

$$Q_{1s} = 1 - \frac{1}{8} \left(\frac{\Phi_s''}{\Phi_s} + \frac{4\pi\rho_0}{\Phi_s} + \frac{eH^2}{2m\Phi_s} \right) \Delta z^2 + \frac{1}{48} \left(\frac{\Phi_s'}{\Phi_s} \right) \left(\frac{\Phi_s''}{\Phi_s} + \frac{4\pi\rho_0}{\Phi_s} + \frac{eH^2}{2m\Phi_s} \right) \Delta z^3. \dots (11)$$

Similar modifications apply to the other coefficients $Q_2 \dots Q_5$.

§ 4. IMAGE ROTATION IN MAGNETIC LENSES

The image rotation in magnetic or combined magnetic and electrostatic lenses can be found easily by numerical step-by-step integration of equation (5). Introducing the series expansions for A and U and considering terms up to the second order in r and r' , one obtains

$$\psi' = \left(\frac{eH^2}{8m\Phi} \right)^{\frac{1}{2}} \left[1 + \frac{r'^2}{2} + \frac{r^2}{8} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} - \frac{H''}{H} \right) \right]. \dots (12)$$

As before, H and Φ are the values of field strength and potential measured on the axis.

As in the case of the radial distance of the trajectory, it is best to express the image rotation in the region of first-order image defects at the point P_n by the rotation ψ_n of a paraxial ray and the deviation $\Delta\psi$ of the rotation of the "first-order" trajectory from the angle ψ_n . The rotation ψ_n of the paraxial ray at the point $z = z_n$ is then given by

$$\psi_n = \int_{z_0}^{z_n} \left(\frac{eH^2}{8m\Phi} \right)^{\frac{1}{2}} dz \quad \dots\dots (13)$$

and the angular first-order deviation $\Delta\psi$ by

$$\Delta\psi_n = \int_{z_0}^{z_n} \left(\frac{eH^2}{8m\Phi} \right)^{\frac{1}{2}} \left[\frac{r'^2}{2} + \frac{1}{8} \left(\frac{\Phi''}{\Phi} + \frac{eH^2}{2m\Phi} - \frac{H''}{H} \right) r^2 \right] dz. \quad \dots\dots (14)$$

The term $\Delta\psi_n$ contains the "anisotropic" image defects (e.g. spiral distortion).

§ 5. TRAJECTORIES THROUGH MAGNETIC LENSES

The expressions for the coefficients appearing in the recursion formulae are rather cumbersome for the combined electrostatic and magnetic lens, but it is now very easy to write down the much simpler expressions to which they reduce when $\Phi \equiv \text{constant}$ or $H \equiv 0$, as is the case in pure magnetic or pure electrostatic lenses.

The magnetic lens will be considered first. The recursion formulae, equations (6), (8) and (9), remain unchanged. Since $\Phi \equiv \text{constant}$, all derivatives of Φ vanish, and the coefficients $Q_1 \dots Q_4$ contain only the expression $eH^2/8m\Phi$, whereas in Q_5 appear also H' and H'' .

Before putting down the simplified equations for $Q_1 \dots Q_5$ it is convenient, particularly with a view to the numerical work, to express the field distribution and the z and r coordinates in relative units rather than in centimetres. As unit of measurement we choose the *radius* of the pole-piece bore (or of the smaller pole piece, in the case of different bore diameters), $R_0 = D/2$. Inspection of the general ray equation (4) shows that if r and z are measured relatively in the unit R_0 , the expression $eH^2/8m\Phi$ is replaced by $eH^2R_0^2/8m\Phi$.

Further, the axial field distribution $H = H(z)$ will be expressed by the ratio of the local field strength to the maximum field strength H_m , or

$$H(z) = H_m h(z). \quad \dots\dots (15)$$

Writing

$$eH_m^2R_0^2/8m\Phi = k^2, \quad \dots\dots (16)$$

where k^2 is a dimensionless constant expressing the strength of the lens excitation* for given operating conditions of the lens, one has $eH^2R_0^2/8m\Phi = k^2h^2$.

If the numerical values of the fundamental constants appearing in equation (16) are inserted, relation (17) results, which is used in actual computations:

$$k^2 = 0.02199 H_m^2 R_0^2 / V, \quad \dots\dots (17)$$

where H_m is measured in oersteds (gauss), V in volts and R_0 in centimetres.

* This constant k^2 is similar to, but not identical with, Glaser's constant $k^2 = eH_m^2 a^2 / 8m\Phi$, as the "half-width" a of Glaser's field is usually not equal to the "half-diameter" R_0 of the pole pieces.

The coefficients $Q_1 \dots Q_5$ then become:

$$Q_1 = 1 - k^2 h^2 (\Delta z^2 / 2), \quad \dots \dots (18a)$$

$$Q_2 = 1 - k^2 h^2 (\Delta z^2 / 6) \Delta z, \quad \dots \dots (18b)$$

$$Q_3 = Q_1, \quad \dots \dots (18c)$$

$$Q_4 = -k^2 h^2 \Delta z, \quad \dots \dots (18d)$$

$$Q_5 = k^2 h^2 \left\{ \left[\frac{1}{2} (h''/h) - \frac{1}{4} k^2 h^2 \right] r_{n+1}^3 + (h'/h) r_{n+1}^2 r'_{n+1} - r_{n+1} r_{n+1}'^2 \right\} \Delta z. \quad \dots \dots (18e)$$

The set of equations (18) is obviously very much simpler than the set of equations (7). In the paraxial case only $k^2 h^2$ appears, and the tracing of a paraxial trajectory can be done very rapidly; calculating the first-order deviations from the paraxial ray is only slightly more involved.

§ 6. TRAJECTORIES THROUGH ELECTROSTATIC LENSES

The recursion formulae (8) and (9) remain unaltered for electrostatic lenses; the coefficients $Q_1 \dots Q_5$ are obtained from equations (7a)–(7d) and (10) by putting $H = H' = H'' = 0$:

$$Q_1 = 1 - \frac{1}{8} \left(\frac{\Phi''}{\Phi} \right) \Delta z^2 + \frac{1}{48} \left(\frac{\Phi''}{\Phi} \right) \left(\frac{\Phi'''}{\Phi} \right) \Delta z^3, \quad \dots \dots (19a)$$

$$Q_2 = \Delta z - \frac{1}{4} \left(\frac{\Phi''}{\Phi} \right) \Delta z^2 + \frac{1}{24} \left[\left(\frac{\Phi''}{\Phi} \right)^2 - \left(\frac{\Phi'''}{\Phi} \right) \right] \Delta z^3, \quad \dots \dots (19b)$$

$$Q_3 = 1 - \frac{1}{2} \left(\frac{\Phi''}{\Phi} \right) \Delta z + \frac{1}{8} \left[\left(\frac{\Phi''}{\Phi} \right)^2 - \left(\frac{\Phi'''}{\Phi} \right) \right] \Delta z^2, \quad \dots \dots (19c)$$

$$Q_4 = \frac{1}{4} \left(\frac{\Phi''}{\Phi} \right) \Delta z + \frac{1}{16} \left(\frac{\Phi''}{\Phi} \right) \left(\frac{\Phi'''}{\Phi} \right) \Delta z^2, \quad \dots \dots (19d)$$

$$Q_5 = \left\{ \left[\frac{1}{32} \left(\frac{\Phi''''}{\Phi} \right) - \frac{1}{16} \left(\frac{\Phi'''}{\Phi} \right)^2 \right] r_n^3 + \frac{1}{8} \left[\left(\frac{\Phi'''}{\Phi} \right) - \left(\frac{\Phi''}{\Phi} \right) \left(\frac{\Phi'''}{\Phi} \right) \right] r_{n'}^2 + \right. \\ \left. - \frac{1}{4} \left(\frac{\Phi''}{\Phi} \right) r_n r_n'^2 - \frac{1}{2} \left(\frac{\Phi''}{\Phi} \right) r_n'^3 \right\} \Delta z. \quad \dots \dots (19e)$$

The choice of the unit in which r and z are measured is immaterial in electrostatic lenses as the same scaling factor enters into both sides of the ray equation: r and z may therefore be measured in centimetres or in relative units, e.g. the lens radius R_0 .

If only the paraxial trajectory is required, Picht's transformation $\rho = r\Phi^{\frac{1}{2}}$, which reduces the paraxial ray equation to $\rho'' = -\frac{3}{10}(\Phi'/\Phi)^2 \rho$, can be used profitably.

The term $\frac{3}{10}(\Phi'/\Phi)^2$ corresponds to the term $k^2 h^2$ of the magnetic lens, and equations (18a)–(18d) can be used as they stand if $\frac{3}{10}(\Phi'/\Phi)^2$ is written instead of $k^2 h^2$. If only the paraxial focal length f_1 and the position of principal planes are required, it is not necessary to compute r for all z values used, since

$$f_1 = \frac{\rho_0}{\rho_1} \left(\frac{\Phi_1}{\Phi_0} \right)^{\frac{1}{2}}. \quad \dots \dots (20)$$

Whilst it would be possible to evaluate a first-order trajectory in the same way, it would not be correct to introduce $\frac{2}{16}(\Phi'/\Phi)^2$ in the place of k^2h^2 in equation (18 e) with corresponding terms instead of (h'/h) and (h''/h) ; the correct expression for the coefficient Q_5 is much more complicated, and the determination of a first-order trajectory or of Δr_n in this way cannot be recommended.

§7. DETERMINATION OF LENS CONSTANTS AND ABERRATIONS FROM TRAJECTORIES

The paraxial lens constants, i.e. the two focal lengths and the positions of the principal planes, are completely determined if two independent trajectories are known. In the case of a lens which is symmetrical with respect to a plane perpendicular to the optical axis, only one ray need be known (if both object and image are not located symmetrically).

Mostly, one will choose at least one ray parallel to the axis outside the lens i.e. $r'_0 = 0$; the trajectory then passes through the image side focus, $r_1 = 0$. The position of the conjugate principal plane is then given as intersection of the straight lines $r'_0 = 0$ and $r' = r'_1$. If the lens is not symmetrical, a second ray parallel to the axis may be traced through the lens from the opposite end, or a ray leaving the object point under a given angle, $r'_0 \neq 0$, is used. However, the choice of the best ray or rays for a particular purpose may depend on what lens property is of interest and on the type of lens.

The five Seidel errors, viz. spherical aberration, coma, astigmatism, curvature, and distortion, are determined by the Δr_1 and $\Delta r'_1$ values for two independent trajectories, yielding four equations for the five unknown aberration coefficients, which, together with the determination of the Petzval curvature from a direct integration of a simple expression given by Glaser (1935, p. 195), are sufficient for the complete evaluation of the coefficients. (The tangential astigmatism can be obtained from a non-skew trajectory; the sagittal astigmatism can then be derived from the knowledge of the tangential astigmatism and of the Petzval curvature. This bears out the statement in §2 that skew rays need not be considered.) If the trajectory represents a ray parallel to the optical axis outside the lens field, Δr_1 will give the spherical aberration and $\Delta r'_1$ the coma of the lens with little further effort.

It should be remembered that the coma and astigmatism obtained in this way would be due to the use of oblique pencils in lenses of perfect rotational symmetry, and should not be confused with the errors arising from imperfections of the rotational lens symmetry. The only case of interest so far where such oblique pencils play a part appears to be the photoelectric image converter (Morton and Ramberg 1936).

Another way of obtaining the aberration constants once the paraxial ray has been found from formulae (8) would be to insert the values r and r' into the integrals representing the aberrations according to Glaser (1935) or Scherzer (1936); but it appears that the building up of Δr with the help of the recursion formulae is rather faster than the required numerical integrations.

Further, some of the integral formulae for the lens aberrations given in the literature are based on integration by parts where use is made of the condition $r_0 = r_1 = 0$ to eliminate certain terms; such integral formulae give, therefore, the aberrations only for the conjugate object and image points, but not for any other point along the trajectory, as is often desired, e.g. in investigations into the connection between field shape and the aberrations arising from it.

Small changes of Φ or H may lead to a small change in lens properties, which can be interpreted as the analogue of chromatic aberration in light optics. In analogy with earlier relations (9), one could again build up the deviation of the new trajectory from the old which is due to a small relative change of electron velocity or potential $\Delta\Phi/\Phi$ or magnetic field $\Delta H/H$, with the help of a set of recurrence formulae. These are more complicated than the recurrence formulae (9); a rather simple integral formula given by Glaser (1940) is preferred. From this the chromatic aberration constants can be calculated easily, provided image and object are situated on the axis and the integration is taken between image and object points.

Frequently, the change of rotation $\Delta\psi_c$ of the image with small changes in voltage or lens excitation current may be of interest in magnetic lenses. This change is easily obtained from equation (13) as

$$\Delta\psi_c = [(\Delta H/H) - \frac{1}{2}(\Delta\Phi/\Phi)]\psi. \quad \dots\dots (21)$$

§ 8. A CHECK OF THE ACCURACY OF THE RAY-TRACING METHOD

During the earlier part of this work, the accuracy of the improved method and its speed were checked against results obtained by other methods for certain assumed field distributions, but finally the numerical results were compared with the exact results given by Glaser (1941) for the "idealized" magnetic lens with the field distribution $H(z) = H_m/[1 + (z/a)^2]$.

Table 1 gives the results of one such comparison for $k^2 = 1$ and $a = 1$, for two slightly different cases. In one case the object was assumed at $z_0 = -3.1$ and a parallel ray was traced from this object ($r'_0 = 0$); the field was still of considerable value at $z = z_0$. In the second case the object was assumed at $z_0 = -15.5$. Here the interval was taken as $\Delta z = 0.5$ from $z_0 = -15.5$ to $z = -5$, as $\Delta z = 0.2$ for $z = -5$ to $z = -3$, and as $\Delta z = 0.1$ from $z = -3$ to $z = +0.8$.

Table 1. Comparison of Numerically Computed and Rigorously Calculated Trajectories

z_0	Lens constant	Exact	Numerical	Error (%)
-3.1	$1/f$	0.829	0.824	-0.60
	z_1	0.777	0.778	+0.13
	C_s	0.786	0.789	+0.32
-15.5	$1/f$	0.797	0.795	-0.26
	z_1	0.761	0.763	+0.26
	C_s	0.742	0.748	+0.90

C_s = spherical aberration constant.

Halving the interval in the second case reduced the errors in $1/f$ from -0.26% to -0.04% , and in z_1 from $+0.26\%$ to -0.08% . This point will be further discussed in the next section, dealing with the factors affecting accuracy.

A further test applied was to see whether a trajectory traced in the reverse direction arrived back at the original starting point. For a trajectory which started at $z_0 = -3.1$, the initial values were $r_0 = 1.0000$, $r'_0 = 0$. The values of the trajectory followed through in the opposite direction were, for $z_1 = -3.1$, $r_1 = 0.99989$ and $r'_1 = -0.00003$, which appears very satisfactory, considering that 38 steps of length $\Delta z = 0.1$ were involved in each direction.

§9. FACTORS AFFECTING ACCURACY

(i) *Choice of Interval*

The best length of interval depends on the accuracy desired, on the shape of the lens field and on the maximum field strength (or refracting power of the lens). From the experience gained so far, a subdivision of the active length of the lens field into $N=20$ to 30 intervals will be required if an accuracy of within 1% or better is desired and the lens is fairly strong ($k^2 \approx 1$, in magnetic lenses). If the lens is weak ($k^2 \approx 0.1$), about 15 to 20 steps will be needed for the stated accuracy. A convenient size of interval will usually be of the order $\Delta z \approx 0.1 R_0$, where R_0 is the radius of the lens bore.

(ii) *Graded Intervals*

If the trajectory is computed for regions where the trajectory is not markedly curved, i.e. where $h^2(z) \ll 1$, it will be permissible to take larger intervals than those mentioned under (i). The length of interval chosen will depend on its position with relation to the main refracting parts of the lens field, being governed by the condition $h^2(z)\Delta z \ll 1$. Frequently it may be advisable to take intervals of several graded lengths, as in the case of the second example given in the preceding section.

No difficulty arises at the junction of a smaller and a larger interval, and the recurrence formulae given can be used without correction when crossing the junction, but, as the interval length Δz enters into the coefficients of the recurrence formulae, some care has to be taken when using graded intervals.

(iii) *Number of Significant Figures Required*

Numerical errors tend to build up in step-by-step processes. By assuming a statistical distribution of small computational errors, one finds that for the total error to be less than a prescribed error δ_0 for N steps of the operation the permissible error δ in each entry must be less than $\delta_0/N\sqrt{3}$.

If the error due to the manipulation of figures is to have negligible influence on the final result, δ_0 should be of the order of 0.001, so that the error in the final result due to all causes is less than 1%. Hence, assuming N to be 20 to 40, say, one finds $\delta \leq 0.00002$; i.e. it will usually be necessary to work to five significant figures. For very accurate work six significant figures will be necessary.

(iv) *Final Errors of the Aberration Constants*

As the coefficient Q_5 , which is the main influence in the building-up of the aberration term, depends on the third power of r and r' , the final error in the spherical aberration constant C_s for instance will be of the order of three times the error in the position of the paraxial trajectory. Hence, if an accuracy of 1% is required in C_s , the values of r and r' along the trajectory would have to be obtained with accuracies of better than 0.3%.

(v) *Required Accuracy of Field Distribution*

Suppose, for instance, that in a magnetic lens with $k^2=1$ and $\Delta z=0.1$, one has $r_n \approx 1$ and $|r'_n| \approx 1$. To obtain the desired error of less than 0.00002 in $Q_1 r_n$ (as derived in (iii)), one would need to know $h(z)$ to approximately one part in 1,000 where $h(z) \approx 1$, and to one part in ten where $h(z)$ is of the order 0.1. However, the accuracy actually required may be about ten times as high in strong lenses

since the most significant coefficient is $Q_4 = -k^2 h^2 \Delta z$, and to obtain the term $Q_4 r_n$ with the suggested accuracy, $h(z)$ would need to be known to approximately one part in 10,000. If the field distribution were known with this high accuracy, one could be confident that only the errors inherent in the step-by-step process are left. This will rarely be the case, and the imperfect knowledge of the field distribution may be the limiting factor in the accuracy with which the lens constants can be evaluated. If one were to suppose that the error due to the inaccuracy in the field determination equals the error due to the computing process, and that the final error in $1/f$ or z_i should be of the order of 1%, one would still need an accuracy of one part in 1,000 where the field is strongest. For lenses which are not very strong ($k^2 \ll 1$) this requirement may be relaxed in proportion to the weakening of the lens, i.e. $\delta \propto 1/k^2$.

The same accuracy desired in the aberration constants would increase the accuracy needed in the field distribution by a factor of three, as discussed in (iv). Fortunately, rapid and very exact experimental field-plotting methods have now been developed (Liebmann 1949 b).

(vi) Estimation of Errors Due to Step-by-Step Process

In a recent paper, Duncan (1948) gave a method of estimating the error (and also of correcting the final result) of a step-by-step integration process from the convergence of the result for several lengths of intervals. It appears that Duncan's index of the set of recurrence formulae used here is $p=3$ for the position r of the trajectory and $p=2$ for the reciprocal focal length r'_i/r_0 . Hence, halving the interval length will improve the accuracy in the position of the trajectory by a factor of 8, and the accuracy in the focal length by a factor of 4, provided computational and field distribution errors can be entirely disregarded.

(vii) Check of Accuracy of Computation

Actual computations should be checked for computational errors and for convergence of the final result.

The simplest way of avoiding computational errors is by using differences. The third and fourth differences should be regular, and the fourth and fifth differences small. It is advisable to make the check by differences a routine part of every computation.

To carry out the convergence test, estimate from the circumstances of the case the probable number of intervals required. Then carry out the computation for this number of intervals, and again for half the number of intervals of double length. From Duncan's test an appreciation of the likely error can be obtained.

(viii) Improvement of Accuracy by Extrapolation

From Duncan's paper a simple extrapolation procedure may be derived, which increases the accuracy of the method by an amount equivalent to doubling the intervals for 50% additional labour. Suppose the computation is first carried out with a large value of Δz , requiring n_1 intervals, giving a final $z_i(1)$ and $r'_i(1)$. Then it is repeated with halved interval length, i.e. with $n_2 = 2n_1$ intervals, giving a final set of values $z_i(2)$ and $r'_i(2)$. Now form the differences $\delta z_i = z_i(2) - z_i(1)$ and $\delta r'_i = r'_i(2) - r'_i(1)$. The final, more correct, value of z_i and r'_i will be

$$z_i = z_i(2) + \frac{1}{2} \delta z_i, \quad \dots \dots (22)$$

$$r'_i = r'_i(2) + \frac{1}{3} \delta r'_i. \quad \dots \dots (23)$$

This procedure is illustrated by a numerical example (Glaser's field, with $z_0 = -3.1$):

$$\begin{aligned} \Delta z_1(1) &= 0.2, n_1 = 20; \Delta z_1(2) = 0.1, n_2 = 40 \\ z_1(1) &= 0.784 & r'_1(1) &= -0.817 \\ z_1(2) &= 0.778 & r'_1(2) &= -0.824 \\ \delta z_1 &= -0.006 & \delta r'_1 &= -0.007 \\ \text{giving } z_1 &= 0.777 & r'_1 &= -0.8263 \\ \text{Exact } z_1 &= 0.777 & r'_1 &= -0.829 \end{aligned}$$

The extrapolation method can be usefully combined in very accurate work with the convergence test by taking, for example, n successively equal to 10, 20 and 40.

§ 10. A SHORT-CUT METHOD

The extrapolation method just described allows the use of a short-cut method of computation if approximate results are required quickly.

Put $Q_1 = 1$, $Q_2 = \Delta z$, $Q_3 = 1$, $Q_4 = -\frac{1}{4}(\Phi''/\Phi + eH^2/2m\Phi)\Delta z$, or, in a purely magnetic lens, $Q_4 = -k^2h^2\Delta z$. Whilst this procedure simplifies the computational work greatly, it reduces the accuracy of the step-by-step method very considerably. However, use of the extrapolation method will make up for much of the loss in accuracy without great increase in labour, since the extrapolation is equivalent to a quadrupling of the final number of intervals. The factors of the differences δz_1 and $\delta r'_1$ will be different for the short-cut process owing to its lower "index", and the corrected values are now given by

$$z_1 = z_1(2) + 0.55 \delta z_1, \quad \dots \dots (24)$$

$$r'_1 = r'_1(2) + \delta r'_1. \quad \dots \dots (25)$$

Applying the short-cut method to the same numerical example as before, one finds (with $\Delta z(1) = 0.2$, $\Delta z(2) = 0.1$):

$$\begin{aligned} z_1(1) &= 0.799 & r'_1(1) &= -0.933 \\ z_1(2) &= 0.782 & r'_1(2) &= -0.881 \\ \delta z_1 &= -0.017 & \delta r'_1 &= +0.052 \\ \text{giving } z_1 &= 0.773 & r'_1 &= 0.829 \\ \text{Exact } z_1 &= 0.777 & r'_1 &= 0.829 \end{aligned}$$

Whilst the excellent agreement of r'_1 computed by the short-cut method with the exact value would seem fortuitous, the example shows clearly the usefulness of the method.

§ 11. CARRYING OUT A COMPUTATION

A computation is best started in this way (e.g. for a magnetic lens): Tabulate the values of $h(z)$, $h'(z)$ and $h''(z)$ for a suitable range of z . Then form and tabulate central values $h_{n+\frac{1}{2}}^2 = h_n h_{n+1}$,

$$\left(\frac{h'}{h}\right)_{n+\frac{1}{2}} = \frac{1}{2} \left(\frac{h'_n}{h_n} + \frac{h'_{n+1}}{h_{n+1}}\right), \text{ and } \left(\frac{h''}{h}\right)_{n+\frac{1}{2}} = \frac{1}{2} \left(\frac{h''_n}{h_n} + \frac{h''_{n+1}}{h_{n+1}}\right).$$

Work out the coefficients $Q_1 \dots Q_4$ from equations (17) and (18 a) to (18 d), and enter these against the z_n values on a new sheet. Then compute r_{n+1} and r'_{n+1} from r_n and r'_n , starting with the given initial r_0 and r'_0 values, using recurrence

formulae (8 a) and (8 b). Continue until the trajectory has arrived at or passed through the image point (e.g. point $r_1=0$), and find the z coordinate of the image point and the slope of the trajectory at this point by interpolation. Check with differences.

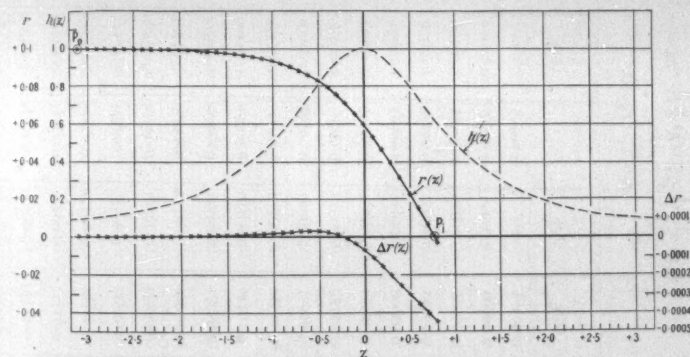
If aberrations are to be computed, calculate the coefficient Q_5 from equation (18 e), with average values of r_n and r'_n (except in the short-cut method), and start computation of Δr_n and $\Delta r'_n$ on a new sheet, beginning with given initial values Δr_0 and $\Delta r'_0$, which will mostly be zero. Interpolate the final value of Δr_1 and $\Delta r'_1$ for $z=z_1$. Check with differences.

Repeat these computations with halved intervals, as discussed in § 9 (vii) and (viii), check with differences, and extrapolate the final results.

§ 12. A SPECIMEN COMPUTATION

A specimen computation, for a magnetic lens with Glaser's field distribution $h(z)=1/(1+z^2)$, and $k^2=1$ is given in Tables 2 and 3 for the initial conditions $r_0=0.1$ and $r'_0=0$ for $z_0=-3.1$, with an interval $\Delta z=0.1$.

The field distribution, the resulting paraxial trajectory and the deviation of the first-order trajectory from the paraxial trajectory are given in the Figure.



Axial magnetic field distribution $h(z)$, computed paraxial trajectory $r(z)$, and aberration $\Delta r(z)$ as a function of z for Glaser's field distribution.

$$(k^2=1, z_0=-3.1, r'_0=0, a=1.)$$

§ 13. CONCLUSION

The method described in this paper was applied to several lenses of known properties and was compared with the other methods mentioned in the introduction. This work indicated that the method presented here is the most rapid so far proposed. Conversely, for given effort, it yields certain information, e.g. the lens aberration constants, more accurately. It is now being applied to the investigation of the aberrations of further types of electron lenses.

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Table 2. Specimen Computation of Paraxial Trajectory

$$\left\{ \begin{aligned} h(z) &= \frac{1}{1+z^2}, \quad k^2=1, \quad z_0 = -3.1, \\ r_0 &= 1, \quad r'_0 = 0. \end{aligned} \right.$$

z_n	h_{n+1}	$Q_n = Q_0$	Q_n	$Q_n r'_n$	r_{n+1}	$Q_n r'_n$	$Q_n r_n$	r'_{n+1}	Differences of r_n			
									$\Delta' r$	$\Delta'' r$	$\Delta''' r$	$\Delta'''' r$
-3.1	0.00943	0.99995	0.10000	0.99995	0	0.00094	0.00094	-0.00094	0.00014			
-3.0	0.0063	0.99995	0.10000	0.99990	0.99981	-0.00094	0.00106	-0.00200	0.00026	0.00012	0.00001	
-2.9	0.01202	0.99994	0.10000	0.99975	0.99955	-0.00020	0.00120	-0.00320	0.00039	13	2	0.00001
-2.8	0.01365	0.99993	0.10000	0.99948	0.99916	-0.00032	0.00136	-0.00456	0.00054	15	1	-1
-2.7	0.01555	0.99992	0.10000	0.99908	0.99862	-0.00046	0.00155	-0.00611	0.00070	16	3	2
-2.6	0.01778	0.99991	0.10000	0.99853	0.99792	-0.00061	0.00178	-0.00789	0.00089	19	3	0
-2.5	0.02040	0.99990	0.10000	0.99782	0.99703	-0.00079	0.00204	-0.00993	0.00111	22	5	2
-2.4	0.02352	0.99988	0.10000	0.99691	0.99592	-0.00099	0.00235	-0.01228	0.00137	27	2	-3
-2.3	0.02722	0.99986	0.10000	0.99578	0.99455	-0.0123	0.00271	-0.01499	0.00166	29	4	2
-2.2	0.03165	0.99984	0.10000	0.99439	0.99289	-0.00150	0.00315	-0.01814	0.00199	33	8	4
-2.1	0.03697	0.99982	0.09999	0.99271	0.99090	-0.00181	0.00367	-0.02181	0.00240	41	6	-2
-2.0	0.04338	0.99978	0.09999	0.99068	0.98850	-0.00218	0.00430	-0.02611	0.00287	47	8	2
-1.9	0.05116	0.99974	0.09999	0.98824	0.98563	-0.00261	0.00506	-0.03116	0.00342	55	9	1
-1.8	0.06063	0.99970	0.09999	0.98533	0.98221	-0.00312	0.00598	-0.03713	0.00406	64	14	5
-1.7	0.07221	0.99964	0.09999	0.98186	0.97815	-0.00371	0.00709	-0.04421	0.00484	78	15	1
-1.6	0.08643	0.99957	0.09999	0.97773	0.97331	-0.00442	0.00845	-0.05264	0.00577	93	18	3
-1.5	0.10395	0.99948	0.09998	0.97280	0.96754	-0.00526	0.01012	-0.06273	0.00688	111	22	4
-1.4	0.12559	0.99937	0.09998	0.96693	0.96066	-0.00627	0.01215	-0.07484	0.00821	133	29	7

-1.3	0.15236	0.99924	0.09998	0.95993	-0.00748	0.95245	-0.07478	0.01463	-0.08941	162	31	2
-1.2	0.18545	0.99907	0.09997	0.95156	-0.00894	0.94262	-0.08933	0.01766	-0.10699	193	40	9
-1.1	0.22625	0.99887	0.09996	0.94155	-0.01069	0.93086	-0.10687	0.02132	-0.12819	233	48	8
-1.0	0.27625	0.99862	0.09995	0.92958	-0.01281	0.91677	-0.12801	0.02570	-0.15371	281	55	7
-0.9	0.33689	0.99832	0.09994	0.91523	-0.01536	0.89987	-0.15345	0.03087	-0.18432	336	61	6
-0.8	0.40923	0.99795	0.09993	0.89803	-0.01842	0.87961	-0.18394	0.03680	-0.22074	397	64	3
-0.7	0.49348	0.99753	0.09992	0.87744	-0.02206	0.85538	-0.22019	0.04337	-0.26356	461	67	3
-0.6	0.58823	0.99706	0.09990	0.85287	-0.02633	0.82654	-0.26279	0.05026	-0.31305	528	58	-9
-0.5	0.68966	0.99655	0.09989	0.82369	-0.03127	0.79242	-0.31197	0.05694	-0.36891	586	42	-16
-0.4	0.79089	0.99604	0.09987	0.78928	-0.03684	0.75244	-0.36745	0.06259	-0.43004	628	18	-24
-0.3	0.88215	0.99559	0.09985	0.74912	-0.04294	0.70618	-0.42814	0.06628	-0.49442	646	12	-30
-0.2	0.95202	0.99524	0.09984	0.70282	-0.04936	0.65346	-0.49207	0.06713	-0.55920	634	46	-34
-0.1	0.99010	0.99505	0.09984	0.65023	-0.05583	0.59440	-0.55643	0.06459	-0.62102	588	74	-28
0	0.99010	0.99505	0.09984	0.59146	-0.06200	0.52946	-0.61795	0.05875	-0.67670	514	92	-18
0.1	0.95202	0.99524	0.09984	0.52694	-0.06756	0.45938	-0.67348	0.05032	-0.72380	422	99	-7
0.2	0.88215	0.99559	0.09985	0.45735	-0.07227	0.38508	-0.72061	0.04046	-0.76107	323	94	5
0.3	0.79089	0.99604	0.09987	0.38356	-0.07601	0.30755	-0.75806	0.03042	-0.78848	229	83	11
0.4	0.68966	0.99655	0.09989	0.30649	-0.07876	0.22773	-0.78576	0.02119	-0.80695	146	65	18
0.5	0.58823	0.99706	0.09990	0.22706	-0.08061	0.14645	-0.80458	0.01338	-0.81796	81	51	14
0.6	0.49348	0.99753	0.09992	0.14609	-0.08173	0.06436	-0.81594	0.00722	-0.82316	30		
0.7	0.40923	0.99795	0.09993	0.06423	-0.08226	-0.01803	-0.82147	0.00263	-0.82410			

$$\pi_i = 0.778; \quad \frac{1}{f} = 0.824$$

Table 3. Specimen Computation of First-Order Aberration

$$\left\{ \begin{aligned} h(z) &= \frac{1}{1+z^2}, \quad k^2=1, \quad z_0 = -3.1, \\ r_0 &= 0.1, \quad r'_0=0, \quad \Delta r_0=0, \quad \Delta r'_0=0. \end{aligned} \right.$$

z_n	Q_1-Q_3	Q_2	Q_4	$Q_4\Delta r_n$	$Q_2\Delta r'_n$	Δr_{n+1}	$Q_3\Delta r'_n$	$Q_4\Delta r_n$	Q_5	$\Delta r'_{n+1}$	Differences of Δr_n		
											$\Delta(\Delta r)$	$\Delta'(\Delta r)$	$\Delta''(\Delta r)$
-3.1	1	0.1		0	0	0	0	0	0	0			
-3.0	1	0.1		0	0	0	0	0	0	0			
-2.9	1	0.1		0	0	0	0	0	0	0			
-2.8	1	0.1		0	0	0	0	0	0	0			
-2.7	1	0.1		0	0	0	0	0	-0.000001	0.000001			
-2.6	1	0.1		0	0	0	0.000001	0	-0.000001	0.000002			
-2.5	1	0.1		0	0	0	0.000002	0	-0.000001	0.000003			
-2.4	1	0.1		0	0	0	0.000003	0	-0.000001	0.000004	0		
-2.3	1	0.1		0	0	0	0.000004	0	-0.000001	0.000005	0		
-2.2	1	0.1		0	0.000001	0.000001	0.000005	0	-0.000001	0.000006	0.000001		
-2.1	1	0.1		0.000001	0.000001	0.000002	0.000006	0	-0.000001	0.000007	0		
-2.0	1	0.1		0.000002	0.000001	0.000003	0.000007	0	-0.000002	0.000009	1		
-1.9	1	0.1		0.000003	0.000001	0.000004	0.000009	0	-0.000002	0.000011	1		
-1.8	1	0.1		0.000004	0.000001	0.000005	0.000011	0	-0.000002	0.000013	1		
-1.7	1	0.1		0.000005	0.000001	0.000006	0.000013	0	-0.000003	0.000016	1	0.000001	
-1.6	1	0.1		0.000006	0.000002	0.000008	0.000016	0	-0.000003	0.000019	2	0	
-1.5	1	0.1		0.000008	0.000002	0.000010	0.000019	0	-0.000003	0.000022	2	0	
-1.4	1	0.1		0.000010	0.000002	0.000012	0.000022	0	-0.000004	0.000026	3	1	

-1.3	1	0.1	0.00012	0.00003	0.00015	0.00026	0	-0.00004	0.00030	0	3	0
-1.2	1	0.1	0.00015	0.00003	0.00018	0.00030	0	-0.00003	0.00033	0	3	0
-1.1	1	0.1	0.00018	0.00003	0.00021	0.00033	0	-0.00002	0.00035	0	4	1
-1.0	1	0.1	0.00021	0.00004	0.00025	0.00035	0.00001	-0.00001	0.00033	0	3	-1
-0.9	1	0.1	0.00025	0.00003	0.00028	0.00033	0.00001	-0.00006	0.00026	0	3	0
-0.8	1	0.1	0.00028	0.00003	0.00031	0.00026	0.00001	-0.00014	0.00011	0	3	0
-0.7	1	0.1	0.00031	0.00001	0.00032	0.00011	0.00002	-0.00025	-0.00016	-0.00001	1	-2
-0.6	1	0.1	0.00032	0.00002	0.00030	0.00016	0.00002	-0.00040	-0.00058	-0.00001	-2	-3
-0.5	1	0.1	0.00030	0.00006	0.00024	0.00058	0.00002	-0.00052	-0.00112	-0.00001	-6	-4
-0.4	1	0.1	0.00024	0.00011	0.00013	0.00012	0.00002	-0.00074	-0.00188	-0.00001	-11	-5
-0.3	0.99559	0.1	0.00013	0.00019	0.00006	0.00018	0.00001	-0.00085	-0.00273	-0.00001	-19	-8
-0.2	0.99524	0.1	0.00006	0.00027	0.00033	0.00027	0.00001	-0.00086	-0.00357	-0.00001	-27	-6
-0.1	0.99505	0.1	0.00033	0.00036	0.00069	0.00035	0.00003	-0.00078	-0.00430	-0.00001	-36	-7
0	0.99505	0.1	0.00069	0.00043	0.00112	0.000428	0.00007	-0.00061	-0.00482	-0.00001	-43	-4
0.1	0.99524	0.1	0.00111	0.00048	0.00159	0.000480	0.00011	-0.00043	-0.00512	-0.00001	-47	-3
0.2	0.99559	0.1	0.00158	0.00051	0.00209	0.000510	0.00014	-0.00028	-0.00524	-0.00001	-50	-1
0.3	0.99604	0.1	0.00208	0.00052	0.00260	0.000522	0.00017	-0.00018	-0.00523	-0.00001	-51	0
0.4	0.99635	0.1	0.00259	0.00052	0.00311	0.000521	0.00018	-0.00011	-0.00514	-0.00001	-51	1
0.5	0.99706	0.1	0.00310	0.00051	0.00361	0.000512	0.00018	-0.00007	-0.00501	-0.00001	-50	0
0.6	0.99753	0.1	0.00360	0.00050	0.00410	0.000500	0.00018	-0.00004	-0.00486	-0.00001	-49	0
0.7	0.99795	0.1	0.00409	0.00049	0.00458	0.000485	0.00017	-0.00002	-0.00470	-0.00001	-48	1
0.8	0.99832	0.1	0.00457	0.00047	0.00504	0.000469	0.00015	0	-0.00454	-0.00001	-46	2

$$\Delta r_3 = -0.000447; \quad \Delta r_4 = -0.000474$$

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Field Correctors for Parabolic Mirrors

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ABSTRACT. Optical systems to correct the oblique aberrations of large parabolic mirrors in astronomical telescopes are discussed. The earlier work of F. E. Ross on doublet lens correctors in the converging beam is extended, and the residual first order spherical aberration is shown to depend only on the power of the corrector, its elimination requiring an impracticably large power; these results are independent of the arrangement and the optical constants of the components, and are unaffected by aspherizing the lens surfaces. Correction by two aspheric plates is shown to lead to systems of relatively great length, with consequent heavy vignetting. Finally, systems are discussed consisting of a doublet lens with one aspheric plate or secondary mirror; such a secondary mirror may be used to correct field curvature, giving corrector lenses which are afocal and therefore free from secondary spectrum, and the departure of the secondary mirror from the spherical form may be made zero by a suitable choice of the other variables. Details are given of the calculated performance of four three-component systems.

§1. INTRODUCTION

A NUMBER of optical systems have been devised giving various degrees of aberration correction for astronomical reflecting telescopes over a substantial angle of field, and a general survey of such systems has been given by F. E. Ross (1935). Most of these systems require a primary mirror which is not parabolic, thus impairing the effective use of the telescope at the prime focus of the mirror alone. Ross has investigated corrector systems for parabolic mirrors consisting of a close pair of thin lenses of substantially zero power in the converging beam, and has shown that such systems can give good correction over a considerable field of coma, astigmatism and chromatic errors, but that spherical aberration cannot then be corrected. This paper describes a more general investigation into corrector systems of pairs of lenses whose combined power is not restricted to zero, and in which a third element is introduced in order to achieve correction of spherical aberration and field curvature as well as coma, astigmatism and chromatic aberrations. A high degree of distortion correction is not normally required, and would generally require systems of greater complexity.

§ 2. DOUBLET CORRECTOR SYSTEMS

A pair of thin lenses each of finite power can only be corrected for axial chromatic aberration if the powers are of opposite sign, and it is well known that such a system will only be simultaneously corrected for field colour if the separation between the lenses is negligibly small. A general indication of the characteristics of possible lens pairs may therefore be obtained by considering the Seidel aberration coefficients for thin lenses in contact; such an investigation will include all possible solutions, though some solutions found in this way may be invalid, due to the presence of large aberrations of higher order. Following the formulation given by Whittaker (1907), the conditions for the removal of the various first order aberrations for systems of spherical surfaces are:

$$\left. \begin{array}{ll} \text{Spherical aberration} & S_I = \Sigma A = 0, \\ \text{Coma} & S_{II} = \Sigma AB = 0, \\ \text{Astigmatism} & S_{III} = \Sigma AB^2 = 0, \\ \text{Field curvature} & S_{IV} = \Sigma P = 0, \\ \text{Distortion} & S_V = \Sigma B(AB^2 + P) = 0, \end{array} \right\} \dots\dots (1)$$

where Σ denotes summation over all the surfaces of the system, and at each surface, i ,

$$A_i = Q_i^2 h_i^4 \left(\frac{1}{\mu_{i-1} s_i} - \frac{1}{\mu_i s_i'} \right), \quad B_i = -\frac{1}{Q_i h_i^2} - \sum_{p=1}^{i-1} \frac{1}{\mu_p h_p h_{p+1}} = -\frac{1}{Q_i h_i^2} - E_i,$$

$$P_i = \frac{1}{r_i} \left(\frac{1}{\mu_i} - \frac{1}{\mu_{i-1}} \right), \quad Q_i = \mu_{i-1} \left(\frac{1}{r_i} - \frac{1}{s_i} \right) = \mu_i \left(\frac{1}{r_i} - \frac{1}{s_i'} \right),$$

r_i being the radius of curvature of the surface, μ_{i-1} and μ_i the refractive indices of the media on each side of it, s_i and s_i' the corresponding conjugate focal distances for a paraxial ray traced through the system with an incident height from the axis h_i , and d the axial distance between successive surfaces. The surface p from which the summation is taken in the equation for B is the stop plane. The factor E_i is the only one involving the stop position and is a simple summation; it follows therefore that if the stop is moved to a new position giving a change of E_i to $E_i - \delta E$, the values of E for all surfaces are changed by the same amount, and denoting the new values of the aberration coefficients by S'_I , S'_{II} etc., it follows from equations (1) that

$$\left. \begin{array}{l} S'_I = S_I, \\ S'_{II} = S_{II} + S_I \delta E, \\ S'_{III} = S_{III} + 2S_{II} \delta E + S_I (\delta E)^2, \\ S'_{IV} = S_{IV}, \\ S'_V = S_V + (S_{IV} + 3S_{III}) \delta E + 3S_{II} (\delta E)^2 + S_I (\delta E)^3. \end{array} \right\} \dots\dots (2)$$

For a parabolic mirror, with radius of curvature at the vertex r_M and semi-aperture h , the spherical aberration for an infinite object distance is zero, so that the coma coefficient is independent of stop position; its value, taking the radius as positive, is $-2h^2/r_M^2$. Since the coma coefficient is finite, the astigmatism coefficient varies with stop position, and is, of course, zero for a stop at a distance $\frac{1}{2}r_M$ in front of the mirror; E at the mirror is then $r_M/2h^2$ (these values correspond to those given, for example, by Schwarzschild, 1905).

Considering now the case of a thin lens in air, it follows from equations (1) by elementary algebra that the coefficient S_{III} with respect to a stop at the lens

($E_t=0$) is equal to minus the power of the lens, $-\phi = -(1/s - 1/s')$, and similarly, for a system of thin lenses in contact, the astigmatism coefficient is equal to minus the total power, $\Sigma(-\phi)$. For a system of thin lenses in contact, at a distance d from the mirror in the converging beam, giving with the parabolic mirror a system free from spherical aberration, coma, and astigmatism, let the aberration coefficients of the lens system be ${}_LS_I$, ${}_LS_{II}$ etc. Then since the parabolic mirror is corrected for spherical aberration alone, so must the lenses be, i.e. ${}_LS_I=0$. The coma coefficient of the lens system will therefore be independent of stop position, and must be equal and opposite to that of the mirror:

$${}_LS_{II} = 2h^2/r_M^2.$$

The astigmatism coefficient of the lens system must be zero for the stop position giving zero astigmatism for the mirror. Since at the lens ${}_Lh = h - 2dh/r_M$, the value of ${}_LE$ for the stop giving $E_M = r_M^2/2h^2$ is

$${}_LE = \frac{r_M}{2h^2} + \frac{d}{h(h-2dh/r_M)} = \frac{r_M^2}{2h^2(r_M-2d)}.$$

Applying equations (2) to the evaluation of ${}_LS_{III}$ for this stop position from its known value when the stop coincides with the lenses, and taking $\delta E = -r_M^2/2h^2(r_M-2d)$,

$${}_LS_{III} = \Sigma(-\phi) - 2{}_LS_{II} \frac{r_M^2}{2h^2(r_M-2d)} = 0,$$

or, substituting for ${}_LS_{II}$ above,

$$\Sigma\phi = -2/(r_M-2d). \quad \dots\dots(3)$$

It follows that the spherical aberration can only be corrected simultaneously with coma and astigmatism by a system of thin lenses in contact in the converging beam if their combined power is such as to give, in combination with the primary mirror, an afocal system, this result being independent of the number of component lenses or the type of glass used. To obtain an image it would be necessary therefore either to add a further optical system behind the corrector lens, or to depart slightly from the condition of spherical aberration correction; apart from the objection of the additional complication of the former alternative, such systems would not be satisfactory, since for the small values of (r_M-2d) necessary to give the corrector lens a small diameter relative to the primary mirror, the power of the corrector lens increases so that secondary spectrum errors become intolerably large. Thus, taking the focal length of the mirror as unit, equation (3) gives for $d=0.90$, $\Sigma\phi = -10.0$ and for $d=0.95$, $\Sigma\phi = -20.0$. For a primary mirror of 100 inches diameter these values of d correspond to lenses having an effective axial aperture of 10 inches and 5 inches respectively. Taking the figures given by Danjou and Couder (1935) for the tolerable limits of secondary spectrum, these diameters require that the lens systems should have focal lengths not less than 280 inches and 70 inches respectively, and hence focal lengths of the primary mirror of 2,800 inches and 1,400 inches. Thus for large telescopes still larger values of d are required to give reasonable secondary spectrum correction with a focal length of the primary mirror which is not impossibly long. But increase of the value of d leads to greater powers of the correcting lens, involving very great uncorrected field curvature. Moreover, these high powers lead to such large lens curvatures that, if a reasonably large field is to be covered, the thin lens approximations are no longer valid, so that axial and oblique colour would no longer be

adequately corrected for a pair of lenses; and for these solutions the higher orders of oblique aberrations become uncontrollably large. It appears therefore that the forms of doublet lenses giving correction of spherical aberration, coma and astigmatism are of no practical value.

It should be noted that the above treatment includes the case where one or more of the surfaces of the thin lenses are aspheric, since the first order central coma and astigmatism for a figured surface are zero, and the equations for their variation with stop position are the same as for spherical surfaced lenses.

If, following Ross, astigmatism and coma alone are corrected, leaving some uncorrected spherical aberration, ${}_LS_I$, and if ${}_LS_{II}$ be the coma with respect to a stop on the lens, then the lens coma with respect to the stop in the incident beam located at the mirror focal plane must be equal and opposite to that of the mirror:

$${}_LS_{II} + {}_LS_I \delta E = 2h^2/r_M^2, \quad \dots\dots (4)$$

and the astigmatism for this stop position must be zero:

$$-\Sigma\phi + 2{}_LS_{II}\delta E + {}_LS_I(\delta E)^2 = 0, \quad \dots\dots (5)$$

whence

$${}_LS_I = \frac{-\Sigma\phi + 4\delta E h^2/r_M^2}{(\delta E)^2}, \quad \dots\dots (6)$$

where, as before, $\delta E = -r_M^2/2h^2(r_M - 2d)$.

The residual spherical aberration for such a system of thin lenses in contact is therefore a linear function of the power and is independent of the type of glass used or the distribution of power between the components. For the case, investigated by Ross, of freedom from secondary spectrum where $\Sigma\phi = 0$,

$${}_LS_I = -8h^4(r_M - 2d)/r_M^4,$$

or, expressing this in terms of S , the spherical aberration of a spherical mirror of the same axial curvature and aperture as the paraboloid, and D , the ratio of d to the focal length of the primary mirror, then $S = -2h^4/r_M^3$ and ${}_LS_I = 4S(1 - D)$. For the case of a lens pair correcting the primary mirror field curvature it follows that the powers of the two lenses, ϕ_1, ϕ_2 , with refractive indices and dispersions n_1, n_2, ν_1, ν_2 , are given, for the flat field condition, by

$$\frac{\phi_1}{n_1} + \frac{\phi_2}{n_2} = \frac{2}{r_M},$$

while for colour correction

$$\frac{\phi_1}{\nu_1} + \frac{\phi_2}{\nu_2} = 0,$$

and the combined power of the doublet lens $\Sigma\phi = \phi_1 + \phi_2$. Equation (6) then gives, after some reduction,

$${}_LS_I = 4S(1 - D) \left\{ 1 + \frac{\nu_1 - \nu_2}{\nu_1/n_1 - \nu_2/n_2} (1 - D) \right\}.$$

The possible range of refractive indices is not large, so that the value of this when $n_1 = n_2 = n$ may be taken as typical. Then ${}_LS_I = 4S(1 - D)\{1 + n(1 - D)\}$, representing an increase, compared with the afocal corrector, of some 15% when $D = 0.9$, and less for larger values of D .

The distortion of the parabolic mirror is negligibly small, so that the distortion of the whole system is effectively equal to that of the lens, which may be evaluated for the general case, using the last of equations (2); for a system of thin lenses in

contact the central distortion is zero, ${}_LS_{IV} = \phi_1/n_1 + \phi_2/n_2$, ${}_LS_{III} = -\Sigma\phi = -\phi_1 - \phi_2$, and ${}_LS_I$, ${}_LS_{II}$ are given by equations (4) and (6), so that

$${}_LS'_V = \left(\frac{\phi_1}{n_1} + \frac{\phi_2}{n_2} - 3\Sigma\phi \right) \delta E + \frac{6h^2}{r_M^2} (\delta E)^2 - 2(-\Sigma\phi + \frac{4h^2}{r_M^2} \delta E) \delta E \\ = \frac{r_M^2}{2h^2(r_M - 2d)} \left[\phi_1 \left(1 - \frac{1}{n_1} \right) + \phi_2 \left(1 - \frac{1}{n_2} \right) - \frac{1}{r_M - 2d} \right]$$

or, expressing ϕ_1 , ϕ_2 in terms of the power $2/r_M$ of the primary mirror as unit, so that $\Phi_1 = \frac{1}{2}\phi_1 r_M$, $\Phi_2 = \frac{1}{2}\phi_2 r_M$,

$${}_LS'_V = \frac{1}{2h^2(1-D)} \left[2\Phi_1 \left(1 - \frac{1}{n_1} \right) + 2\Phi_2 \left(1 - \frac{1}{n_2} \right) - \frac{1}{1-D} \right]. \quad \dots (7)$$

For the case investigated by Ross where $n_1 = n_2$, $\phi_1 + \phi_2 = 0$, ${}_LS'_V = -1/2h^2(1-D)^2$, so that the distortion is independent of the type of glass used and the distribution of power between the components, and increases with increasing D , i.e. it is greater for the forms having lower spherical aberration.

It follows from equation (7) that for the flat field form, over the practically useful range of D , the distortion is slightly lower than for afocal correctors, and again it increases with D .

The treatment given so far has been for the case where the stop is in the parallel incident beam at the front focus of the mirror, which applies to systems to be discussed later. When spherical aberration, coma and astigmatism are all corrected, then they remain so for any stop position, and the distortion is unchanged by stop changes; but with a lens system of the type discussed above having some residual spherical aberration, the coma and astigmatism change with stop position, and when such a system is used without additional complications to remove the spherical aberration, the effective stop would of course be the parabolic mirror itself. The astigmatic coefficient for the parabolic mirror is then $-2/r_M$, so that the amount of the uncorrected spherical aberration may be derived from equation (4) above, which is unaltered, and in place of equation (5)

$$-\Sigma\phi + 2{}_LS_{II}\delta E + {}_LS_I(\delta E)^2 = 2/r_M,$$

the value of δE corresponding to a change of stop position from the lens to the mirror being $-d/h^2(1-2d/r_M)$. Evaluating ${}_LS_I$,

$${}_LS_I = -\frac{8h^4(1-D)^2}{r_M^3 D^2} \left\{ \Sigma\Phi + \frac{1}{1-D} \right\} = 4S \frac{(1-D)^2}{D^2} \left\{ \Sigma\Phi + \frac{1}{1-D} \right\}$$

which for afocal correctors ($\Sigma\Phi = 0$) reduces to

$${}_LS_I = 4S(1-D)/D^2.$$

The spherical aberration in this case is therefore somewhat greater than ($1/D^2$ times) that with the stop in the front focal plane, decreasing as before as D increases*.

* The values for the residual aberration obtained by Ross are intermediate between those obtained here for the case of the stop on the lens and in the focal plane (his ${}_LS_I$ is $4S(1-D)/D$); this is due to the fact that he neglected the astigmatism of the primary mirror in the case where the stop is at the mirror. This gives a slightly reduced value of the spherical aberration, but is only justified for fairly small fields; for example, the separation of the s and t focal lines at the edge of a 12-inch square plate at the focus of a 100-inch $f/5$ parabolic mirror is 0.14 inch.

A similar treatment gives for the distortion when the stop is on the primary mirror

$${}_LS'_V = \frac{D}{2h^2(1-D)} \left[2\Sigma\Phi \left(1 - \frac{1}{n} \right) - \frac{4-3D}{1-D} \right]$$

or, for afocal correctors, where $\Sigma\Phi(1-1/n)=0$,

$${}_LS'_V = -D(4-3D)/2h^2(1-D)^2,$$

giving again, for the same value of D , larger values than with the stop at the front focus for values of $D > \frac{1}{2}$, and greater distortion for larger values of D .

Thus in general, for a given total power of lens system correcting coma and astigmatism, the spherical aberration is less for correctors designed for use further from the mirror, for a given position of corrector lens system the spherical aberration decreases as the total lens power is reduced, and in each case distortion increases as spherical aberration decreases. This result, again, is unaffected by "figuring" any of the lens surfaces, so that no improvement of the first order aberrations can be achieved in this way.

The treatment given in this section includes only the first order aberrations; the higher orders are determined by ray tracing, and in general it is found that as the separation of the lens from the mirror is increased to reduce the spherical aberration, forms of the lens result having deeper curves and larger higher order oblique aberrations. These higher order aberrations of course increase with field angle, so that for larger field sizes the size of corrector lens increases both directly on account of the increased field to be covered and because lenses farther removed from the focal plane must be used.

§ 3. ASPHERIC PLATE CORRECTORS

The correction of the aberrations of a parabolic mirror by a pair of lenses discussed in the previous section is attended by the limitation that for chromatic correction the lenses must be close together. Aspheric plates would not be subject to this restriction, since they are free from first order chromatic aberrations, so that it appears desirable to investigate possible corrector systems consisting of two separated aspheric plates in the converging beam. If A_1, A_2 be the Seidel spherical aberration of two plates, at distances d_1, d_1+d_2 , from the mirror, then the conditions for the removal of coma and astigmatism of the whole system for a stop at the mirror focal plane are

$$A_1E_1 + A_2E_2 = 2h^2/r_M, \quad \dots\dots\dots (8)$$

$$A_1E_1^2 + A_2E_2^2 = 0. \quad \dots\dots\dots (9)$$

The values for the two plates, as defined above, are

$$E_1 = \frac{r_M^2}{2h^2(r_M - 2d_1)}; \quad E_2 = \frac{r_M^2}{2h^2(r_M - 2d_1 - 2d_2)}.$$

From equations (8) and (9) it follows that the spherical aberration of the system is given by

$$A_1 = -\frac{2h^2}{r_M^2} \cdot \frac{E_2}{(E_1 - E_2)E_1}, \quad A_2 = +\frac{2h^2}{r_M^2} \cdot \frac{E_1}{(E_1 - E_2)E_2},$$

$$A_1 + A_2 = \frac{2h^2}{r_M^2} \cdot \frac{E_1 + E_2}{E_1E_2},$$

which can only be zero if E_1 , E_2 have opposite signs, i.e. if one plate is situated in front of the stop at the mirror prime focus. Inserting the values of E_1 , E_2 and, as before, expressing d_1 , d_2 in terms of the mirror focus as unit, and spherical aberration in terms of that of a hypothetical spherical primary S ,

$$A_1 + A_2 = 8h^4(r_M - d_1 - d_2)/r_M^4 = 4S(1 - D_1 - \frac{1}{2}D_2).$$

The corresponding distortion coefficient is

$$A_1E_1^3 + A_2E_2^3 = -[2h^2(1 - D_1)(1 - D_1 - D_2)]^{-1}.$$

Thus for a pair of plates in the converging beam, correcting coma and astigmatism, the total uncorrected spherical aberration depends only on the distance from the mirror of the point midway between the plates, and becomes smaller as this distance approaches the focal length of the mirror.

The spherical aberration coefficients of the individual surfaces are of opposite signs, and for a given position of their mid-point they increase in absolute magnitude as the distance between the plates is reduced; moreover, for a given separation D_2 between the plates, the absolute magnitude of the figuring on the two plates increases as D_1 is increased, and the absolute magnitude of the figuring on the rear plate increases as it approaches the focal plane. Now if the mid-point between the plates is moved further from the mirror to reduce the total uncorrected spherical aberration, the possible separation between the plates is necessarily reduced, and with a given separation the rear plate approaches more closely to the focal plane, so that the individual figurings are increased on both counts, introducing uncorrectable higher order field aberrations for positions where the spherical aberration is sufficiently reduced to be neglected; such systems of two figured plates therefore appear to offer no advantages over spherical surfaced lens correctors. As with lens systems, the distortion increases as separations are chosen to give lower spherical aberration; first-order field curvature cannot be corrected, since the Petzval sum for figured plates is zero.

A similar treatment of the case of two Schmidt plates in the parallel beam in front of the mirror shows that spherical aberration coma and astigmatism may then be corrected by the use of two plates so placed that the sum of their distances from the mirror is twice the focal length of the mirror, having equal and opposite figuring which increases in magnitude as the two plates approach each other. With one plate in the converging beam and one in the parallel beam, the latter must have a distance from the mirror of more than twice the mirror focal length, this distance increasing as the second plate is moved nearer to the focal plane. The length of all such systems is therefore very considerable, and thus produces rapid vignetting of the aperture for off-axis imagery unless the figured plates are considerably larger in diameter than the primary mirror.

§ 4. THREE-ELEMENT CORRECTORS

It is clear from the foregoing that three or more correcting elements are required to give, with a parabolic mirror, an image free from spherical aberration, coma, astigmatism, field curvature and chromatic errors.

The simplest three-element system from the design standpoint, which has been suggested by J. G. Baker (private communication), consists of a doublet lens in the converging beam of the type discussed above, giving correction of all aberrations except spherical aberration and distortion with respect to a stop in

the mirror front focal plane, together with a figured plate in this plane; such a plate may be figured to correct the spherical aberration of the lens, and is free from all other first order aberrations. The form of the fourth power figuring on the plate is given directly from the spherical aberration of the lens system derived in § 2. The retardation of the edge of the wave front is $-\frac{1}{8}S_I$, so that if x be the excess of glass thickness at the margin of the plate over that at the centre given by the equation of the plate profile, $x = ah^4$, the retardation of the wave front produced by the plate is $x(n-1)$; hence

$$\begin{aligned}x(n-1) &= ah^4(n-1) = \frac{1}{8}S_I \\ a &= S_I/8h^4(n-1).\end{aligned}$$

It would be more convenient to move the plate forward slightly to approximately the same plane as the lens (Figure 1). This would require slightly different plate and lens forms, which may be readily calculated using the methods already given. However, the change is small, and in actual design, after the form of thin lens

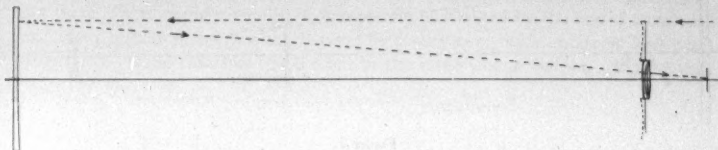


Figure 1.

giving the required aberrations has been found from thin lens theory, small modifications have to be made, based on surface-by-surface Seidel calculations and ray tracing, to allow for the effect of finite thicknesses and separations of the lenses, and to introduce small first order aberration coefficients to give the best compromise with the small higher order aberrations which may be found by ray tracing. The effect of a small movement of the plate may therefore well be dealt with at the same time; and it is desirable, as with the Schmidt system, to give the plate a small vertex power to minimize its chromatic effect. Details of the form and performance of such a system are given in § 5 (example 1).

The disadvantages of such a system are: firstly, the difficulty of making and mounting such a large figured plate, and secondly, the very considerable vignetting that is necessarily produced by such a large separation of mirror and plate. Thus, to secure uniform illumination over a 12-inch square plate with an $f/5$ mirror of 100 inches diameter, the stop in the plate plane, and consequently the effective aperture of the mirror, would be only 82 inches in diameter. This effect could be reduced if the plate could be placed nearer to the mirror, but in intermediate positions it would interfere with the converging beam from the mirror to the lens. The vignetting could be eliminated by using a Ross corrector designed for a stop at the mirror, together with a figured plate in contact with the mirror. Light would pass twice through such a plate, with consequent increased light loss, but this would be much more than off-set by the reduction of vignetting*.

* Placing a figured plate in contact with the mirror is of course substantially equivalent optically to giving the mirror itself a non-parabolic form. A mirror could so be figured as to give, with a designed Ross corrector, simultaneous correction of spherical aberration, coma and astigmatism, but the mirror would not then be corrected for use alone at its prime focus.

It therefore appears desirable to investigate the possibility of correction using a figured plate in the converging beam between the mirror and lens (Figure 2). If d_1 , d_2 be the distances from the mirror of the plate and the lens system respectively, ${}_PS_I$ the spherical aberration of the plate, ${}_LS_I$, ${}_LS_{II}$ etc. the Seidel coefficients of the lens with respect to a stop on the lens, and ϕ its power, then for spherical aberration correction ${}_PS_I + {}_LS_I = 0$.

With respect to a stop on the mirror, the Seidel coma and astigmatism of the plate are ${}_PS_I E_1$ and ${}_PS_I E_1^2$ respectively, where $E_1 = -d_1/\{h^2(1 - 2d_1/r_M)\}$, so that proceeding as before, for coma correction

$${}_PS_I E_1 + {}_LS_{II} + {}_LS_I E_2 = 2h^2/r_M^2$$

and for astigmatic correction

$${}_PS_I E_1^2 - \phi + 2{}_LS_{II} E_2 + {}_LS_I E_2^2 = 2/r_M$$

where $E_2 = -d_2/h^2(1 - 2d_2/r_M)$ (since the figured plate is paraxially afocal).

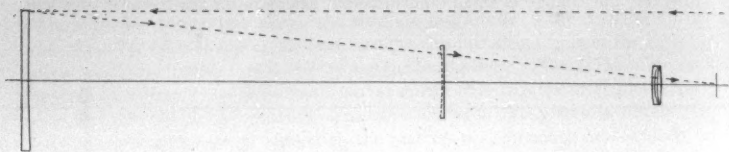


Figure 2.

Eliminating ${}_LS_{II}$ and, as before, expressing the separations in terms of the focal length of the primary mirror ($d_1 = \frac{1}{2}D_1 r_M$, $d_2 = \frac{1}{2}D_2 r_M$) and the power in terms of that of the primary mirror ($\phi = 2\Phi/r_M$),

$$\begin{aligned} {}_LS_I &= -{}_PS_I = -\frac{8h^4}{r_M^3} \frac{(1-D_1)^2(1-D_2)^2}{(D_1-D_2)^2} \left\{ \Phi - \frac{1}{1-D_1} \right\} \\ &= 4S \frac{(1-D_1)^2(1-D_2)^2}{(D_1-D_2)^2} \left\{ \Phi - \frac{1}{1-D_1} \right\}, \end{aligned}$$

giving the required figuring on the plate. The figuring increases rapidly as the separation between plate and lens is decreased, $(D_1 - D_2)$ becoming infinite when the two coincide: this corresponds to the fact found earlier that no solution is obtainable by figuring the thin lens. The figuring increases slightly, as before, for positive values of Φ necessary to give a flat field, and again, as before, lens systems further from the mirror require a smaller figuring, but lead to deeper curves and larger high-order oblique aberrations. The coma coefficient of the plate is of the same sign as that of the mirror, so that the first order coma of the lens pair is greater than in the case of the plate near the front focal plane, leading to somewhat greater curvatures and higher order aberrations. This type of design can therefore only give good results when the plate is rather far removed from the lens and, therefore, rather large, giving a substantial central construction; but the total useful aperture with a 100-inch mirror on a plate 12 in. square is considerably more than in the previous case for uniform illumination. Details of a system of this form are given in example 2 below.

A system having exactly the same first order aberration characteristics and the same lens form would result if the aspheric plate were replaced by a plane

mirror carrying the corresponding figuring (i.e. the constant a in the plate profile equation would be reduced by a factor $2/(n-1)$, n being the plate index). This would have some advantages in compactness (Figure 2(a)).

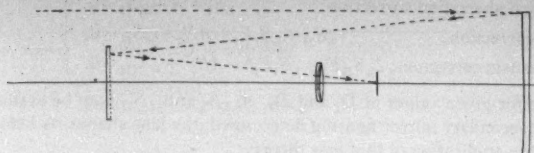


Figure 2 (a).

In the correctors discussed above, the Petzval field curvature is corrected by the lens system, which must consequently have a positive power, and be made up of two glasses having different dispersions, and therefore show some secondary spectrum. In the examples given, this secondary spectrum is very substantially below what are generally regarded as tolerable limits in refracting telescopes, but these effects could virtually be completely eliminated by the use of a convex secondary mirror to correct the Petzval field curvature, with an afocal lens pair (Figure 3). This case may be treated in the same way as before. If r be the

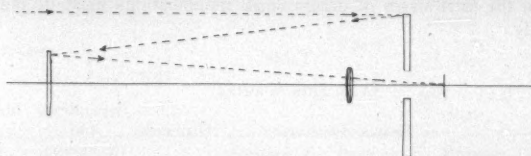


Figure 3.

axial radius of each mirror, with a separation in terms of the primary mirror focus of D_1 , and if the primary mirror be a paraboloid and the secondary mirror spherical, substitution of the appropriate values in equation (1) gives for the pair of mirrors,

$$\text{spherical aberration: } {}_M S_I = (2h^4/r^3)(1 - D_1)^2(1 + D_1)^2,$$

and with respect to a stop on the paraboloid,

$$\text{coma: } {}_M S_{II} = -(2h^3/r^2)[1 - \frac{1}{2}(1 - D_1)^2(1 + D_1)(2 + D_1)],$$

$$\text{astigmatism: } {}_M S_{III} = -(2/r)[1 - \frac{1}{4}(1 - D_1)^2(2 + D_1)^2].$$

If A be the spherical aberration due to figuring of the secondary mirror, then for a stop on the primary the corresponding coma is AE_1 and astigmatism AE_1^2 , where $E_1 = D_1 r / 2h^2(1 - D_1)$. The value of E at the lens pair, E_L , is slightly more complicated than before, since the secondary mirror power affects the incident height h_L on the lens; if d_2 be the separation of the lens from the secondary mirror,

$$E_L = E_1 + d_2/h_1 h_L$$

which reduces to

$$E_L = \frac{r(D_1 + D_2 + D_1 D_2)}{2h^2(1 - D_1 - D_1 D_2)}.$$

If ${}_L S_I$ be the spherical aberration coefficient of the lens system and ${}_L S_{II}$ its coma with respect to a stop on the lens, then since its power is zero its central astigmatism is zero, and, proceeding as before,

$$\left. \begin{array}{l} \text{for spherical aberration correction: } {}_L S_I + A + {}_M S_I = 0, \\ \text{for coma correction: } {}_L S_{II} + {}_L S_I E_2 + A E_1 + {}_M S_{II} = 0, \\ \text{for astigmatism correction: } 2{}_L S_{II} E_2 + {}_L S_I E_2^2 + A E_1^2 + {}_M S_{III} = 0. \end{array} \right\} \dots \dots (10)$$

Hence for given values of D_1 and D_2 , A , ${}_L S_I$ and ${}_L S_{II}$ may be evaluated, and hence the secondary mirror figuring determined, the lens shapes, as before, being given by the application of thin lens theory.

Alternatively, the mirror figuring may be given an arbitrary value, for example zero, and equations (10) then solved to give D_2 in terms of D_1 ; for zero figuring

$$D_2 = 2(1 - D_1) \frac{D_1^2 + [D_1(1 + D_1 - D_1^2)]^{\frac{1}{2}}}{(1 + D_1 - D_1^2 + D_1^3) + 2D_1[D_1(1 + D_1 - D_1^2)]^{\frac{1}{2}}} \dots \dots (11)$$

The advantage of a strictly spherical secondary mirror is not only that it is easier to make, but that accurate centring (as distinct from alignment) is not necessary. These equations are only accurate for thin lenses, and for zero first order aberrations, whereas in fact the higher order aberrations which are present require the introduction of small first order terms to give an optimum compromise, so that in the final stages of design small modifications must be made, semi-empirically.

Table
(All lengths in inches)

	Radius of curvature	Diameter	Separations and thicknesses	Mean refractive index and dispersion
Example 1	Aspheric plate $\left\{ \begin{array}{l} \text{Profile } x = +5.81 \times 10^{-7} h^2 \\ -2.136 \times 10^{-10} h^4 \end{array} \right.$	82.0		$\left\{ \begin{array}{l} 1.519 \\ 60.4 \end{array} \right.$
	Primary mirror	+1000" axial (paraboloid)	450.0	
	Doublet lens $\left\{ \begin{array}{l} 76.92_5 \\ 42.83 \\ 78.12_5 \\ -240.60 \end{array} \right.$	27.0	1.5 1.0 2.0 40.9 to focal plane	$\left\{ \begin{array}{l} 1.579 \\ 43.0 \\ 1.519 \\ 60.4 \end{array} \right.$
	Primary mirror	+1000" axial (paraboloid)	300.0	
Example 2	Figured plate $\left\{ \begin{array}{l} \text{Profile } x = -1.515 \times 10^{-8} h^4 \\ +9.792 \times 10^{-16} h^8, -15820 \end{array} \right.$	51.7	1.0 148.4	$\left\{ \begin{array}{l} 1.517 \\ 60.6 \end{array} \right.$
	Doublet lens $\left\{ \begin{array}{l} 74.23 \\ 37.32 \\ 108.70 \\ -102.86_5 \end{array} \right.$	27.5	1.5 1.5 2.0 40.4 to focal plane	$\left\{ \begin{array}{l} 1.579 \\ 42.6 \\ 1.572 \\ 57.7 \end{array} \right.$

Table (cont.)
(All lengths in inches)

	Radius of curvature	Diameter	Separations and thicknesses	Mean refractive index and dispersion	
Example 3	Primary mirror	-853.3 axial (paraboloid)	100.0		
			256.0		
	Secondary mirror	$\begin{cases} -853.3 \text{ axial} \\ \text{Profile } x = -5.860 \times 10^{-4} h^2 \\ +1.814 \times 10^{-9} h^4 \end{cases}$	46.2	213.3	
	Doublet lens	$\begin{cases} 92.25 \\ 49.44 \\ 1255.0 \\ -112.47 \end{cases}$	24.3	0.85 1.58 1.49	1.572 57.7 1.572 57.7
			68.1 to focal plane		
Example 4	Primary mirror	-1000.0 axial (paraboloid)	100.0		
			350.0		
	Secondary mirror	$\begin{cases} -1000.0 \text{ axial} \\ \text{Profile } x = -5.0 \times 10^{-4} h^2 \\ -7.47 \times 10^{-10} h^4 \end{cases}$	38.4	142.87	
	Doublet lens	$\begin{cases} 81.76 \\ 46.29 \\ -557.75 \\ -87.98 \end{cases}$	24.1	1.0 1.75 1.25	1.572 57.7 1.572 57.7
			68.5 to focal plane		
Relative aperture of primary mirror		Example 1	Example 2	Example 3	Example 4
Equivalent focal length of complete system		f/5.0	f/5.0	f/4.27	f/5.0
Total angle of field to corners of 12 x 12 in. plate		439.6 in.	434.3 in.	705 in.	710 in.
Maximum distortion at corner of 12 x 12 in. plate		2° 08' 50"	2° 13' 00"	1° 22' 18"	1° 21' 40"
Fraction of primary mirror area unobstructed		0.048 in.	0.090 in.	0.048 in.	0.050 in.
Total length of optical system		60.0%	73.3%	78.7%	85.2%
		495.4 in.	494.8 in.	285.3 in.	350.0 in.

Details are given in examples 3 and 4 of two such systems using a secondary mirror of the same axial curvature as the primary, with substantially afocal corrector lens pairs; the secondary mirror figuring in each case is fairly small (63% and 15% respectively of the figuring of the corresponding Cassegrain secondary) and is of opposite sign in the two examples; small changes from example 4 would clearly give a form with an exactly spherical secondary mirror.

It has been shown above that correctors cannot be designed consisting of three or more thin lenses in contact but, using three separated lenses, forms can be found giving correction of the first order aberrations, including chromatic errors, provided the two outer lenses have powers of the same sign as each other, and opposite to that of the central lens. Such systems are not readily amenable

to a general analytical treatment; a more empirical investigation shows that for spherical surfaced lenses very large powers and deep curvatures (with consequent large higher order aberrations) can only be avoided if the lens separations from each other and from the focal plane are so great as to require the lens nearest to the mirror to have a diameter impracticably large for use with large telescopes. This difficulty could to some extent be overcome by aspherizing one or more of the lens surfaces, but such systems do not appear to have any advantage over those described above.

§ 5. NUMERICAL EXAMPLES

The four examples given below have been worked out in each case for a paraboloidal mirror of 100 inches diameter and for a photographic plate 12 inches square. (Designs for smaller fields would require correctors of correspondingly smaller diameters.) In the first two cases, since the field curvature is corrected by a lens system of suitable positive power, the combined focal length is slightly less than that of the primary mirror, so that for a given relative aperture of mirror the field angle at the corner of a plate 12 inches square is correspondingly increased. The use of a convex secondary mirror to correct field curvature (examples 3 and 4) gives a combined focal length greater than that of the primary, with a corresponding increase of the plate scale. The numerical data for the four examples are shown

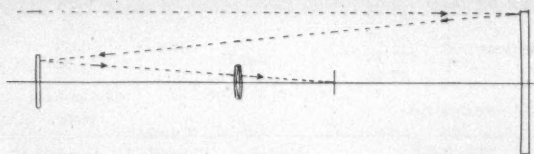


Figure 4.

in the Table. In each case the corrector lens has been taken in such a position that its effective axial aperture is 10 inches. Designs of the same relative aperture, but having the lens nearer to the focal plane, and therefore of smaller diameter, could be used at the cost of some deterioration of the field corrections, or of some reduction of field size. Reduction of the relative aperture of the primary, giving a reduction of field angle for a given plate size, would lead in each case to designs having, for the same degree of correction, lens systems somewhat nearer to the focal plane and therefore smaller. For all these corrector lenses there is a slight reduction of the higher order oblique aberrations as the refractive index of the glasses is increased, since the curvatures are thereby reduced. In examples 1, 3 and 4 (Figures 1, 3 and 4) only fourth-power figuring has been considered, since this gives an axial correction in each case within the Rayleigh limit. In example 2 (Figure 2) an eighth-power term is included in order to give improvement of the oblique imagery while preserving the axial correction.

Figure 5 shows the angular aberrations for d light in a tangential plane over the aperture for points at 6 inches and $8\frac{1}{2}$ inches from the plate centre (centre of plate edges, and plate corner). Numerical integration shows that for examples 1, 3 and 4 over the whole field, and for example 2 up to the plate edge (12-inch diameter circle) these aberrations amount to less than two wavelengths deformation of the wave front. The oblique aberrations in example 2 in the outer

parts of the field are rather greater than is desirable; these would be improved in a design using a primary mirror of greater focal length, but it would probably not be convenient to increase the primary tube length beyond the present 500 inches. Apart from this better correction, examples 3 and 4 have the advantage of lower central obstruction and shorter length, with a reduction of the difficulties arising from tube flexure; the optimum solution appears to lie between these two examples, where the secondary mirror figuring becomes zero. If a smaller field were to be covered, the same degree of correction could be achieved using a smaller lens system nearer to the image plane, and in this case the position of the secondary mirror for zero asphericity, given by equation (11), is further from the primary and, therefore, also reduced in size.

Figure 6 shows the departures of the C, F and g wave fronts from the d wave front for the same two positions in the field, and for the axis of the first two examples where some secondary spectrum is present; the second two examples, without

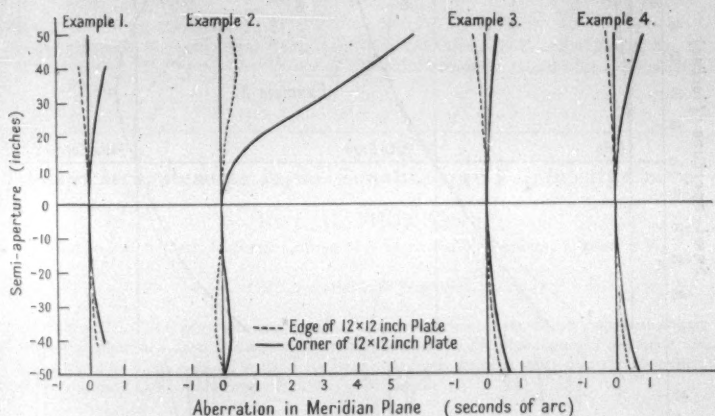


Figure 5.

secondary spectrum, have axial colour errors amounting to about one half wavelength for the whole visible spectral range. All show a small transverse chromatic effect (chromatic difference of distortion) which, as mentioned before, necessarily arises from the finite thickness of the lenses, and decreases at the larger field angles where higher order effects of opposite sign appear. The maximum chromatic image spread for the range C to g is in each case about 0.002 inch. This could be slightly reduced if the axial chromatic correction were allowed to deteriorate somewhat.

Ross (1935) quotes results obtained by Dr. W. Baade showing that the minimum diameter of an axial star image with normal photographic emulsions and the best conditions of seeing is 1.2 seconds of arc for the Mount Wilson 100-inch and other mirrors of well-known excellence. This limit is presumably set by the seeing, but since for a 100-inch $f/5$ mirror it corresponds geometrically to a linear diameter on the plate of 0.003 inch, it is approaching the limit set by photographic grain size, and conditions of perfect seeing could not be expected

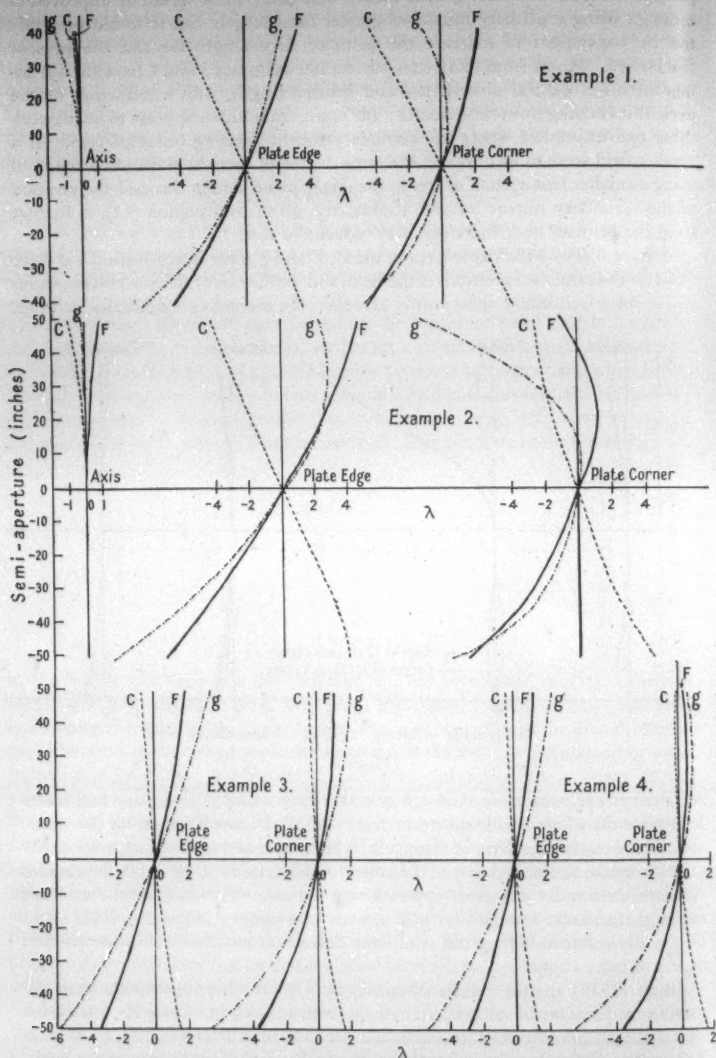


Figure 6. Aberration of C, F and g wavefronts with respect to the d wave front, expressed in d wavelengths.

to yield any greatly increased resolution. Moreover, angular aberrations of this order correspond to a wave front deformation of only very few wavelengths, for which the diameter of the diffraction pattern in the image plane is much less than that of the conventional "disc of confusion" given by geometrical optics. The practical limits of perfection are therefore approached by examples 1, 3 and 4 over the whole field, and by example 2 over a circle 12 inches in diameter.

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Shape Irregularities in the Equal Energy Luminosity Curve

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ABSTRACT. The photopic luminosity function of two observers with normal colour vision has been measured. Two shoulders on the curve in the blue region of the spectrum first noticed by Gibson and Tyndall have been confirmed, and the relationship of the curves to the standard C.I.E. luminosity function is discussed.

§1. INTRODUCTION

FOLLOWING a recent check of the relative energy content of the spectrum of the Wright colorimeter at Imperial College, certain irregularities were noticed in the shape of the normal photopic luminosity function. These irregularities were similar to those reported by Gibson and Tyndall (1923-24), which appeared as two shoulders upon the curve, one at wavelength 440 m μ and the other at 470 m μ , these shoulders representing a local increase in relative sensitivity.

In view of the scant attention given to the meaning of these shoulders, even by Gibson and Tyndall themselves, and because the mean values in the blue and violet spectrum for all their observers differ somewhat from the standard C.I.E. photopic luminosity function, a further determination of the normal photopic curve seemed of interest.

An entirely new calibration of the relative energy content of the spectrum of the Wright colorimeter was therefore made with the help of a R.C.A. 931a multiplier photocell, and the photopic luminosity function of two observers with normal colour vision redetermined.

§2. METHOD

The measurements were made with the Wright colorimeter at Imperial College, the field of which was a square subtending $1^{\circ} 20'$ at the eye. This square was divided horizontally into two equal parts forming an upper test and lower comparison patch. Either patch could be illuminated independently by monochromatic or standard white light of any intensity. The surround field was at zero luminance, and the test patch was viewed through a 2 mm. artificial pupil. The standard white source was provided by an under-run tungsten projection lamp. In other respects the apparatus was similar to that described by Wright (1946) with the addition of the chromatizing lens described by Thomson and Wright (1947).

(i) Calibration of the Apparatus

(a) *Wavelength.* The wavelength of the monochromatic light passing the exit pupil of the colorimeter was determined with a mercury arc lamp and lithium flame, as described previously (Wright 1946).

(b) *Intensity control.* The intensity of the monochromatic light was controlled by a neutral wedge and filters, and for the calibration of this control a R.C.A. 931a photocell was used.

The linearity of the wedge was determined at wavelength $530\text{ m}\mu$ by comparing the reduction of light obtained by a movement of the wedge with that obtained by rotating in the monochromatic beam black sectors of known angle. The comparison was made by equating the photocurrents derived from the cell in each case. The relationship between the position of the wedge and the logarithm of the light intensity was found to be linear within $\pm 2\%$ throughout the whole extent of the wedge.

Since the light stimulus was in one case intermittent and in the other continuous, it was possible that the equation given by the photocell might not agree with that obtained by using the eye as the comparator. In the first instance, therefore, a visual calibration was undertaken, which agreed with that obtained by the photocell method to within the stated limits, both as to linearity and as to the slope of the line relating wedge position to the logarithm of the light intensity. The advantage of the photocell method is that many fewer determinations are needed to achieve the same precision.

The density of the neutral wedge was then determined by the photocell method at each $10\text{ m}\mu$ of wavelength throughout the spectrum from 400 to $700\text{ m}\mu$.

(c) *Spectral energy content (relative).* The sensitivity of the photocell cathode to light of various wavelengths was determined at the National Physical Laboratory. This calibration was performed with a constant output from the cell of $0.3\text{ }\mu\text{a.}$ at all wavelengths.

In using the cell to calibrate the spectrum of the colorimeter, the wavelength $520\text{ m}\mu$ was chosen as a reference, and the intensity so adjusted that a small standard output current (approximately $0.3\text{ }\mu\text{a.}$) was delivered by the cell. The setting of the intensity control was then noted. Another wavelength was chosen, and the new setting of the intensity control required to give the standard photocurrent determined. Following an estimation at one further wavelength, the intensity setting for the standard wavelength $520\text{ m}\mu$ was redetermined. The wavelength settings, which were made at every $20\text{ m}\mu$ from 400 to $700\text{ m}\mu$, were taken in random order with a determination of the standard wavelength, $520\text{ m}\mu$ between each pair.

From the settings of the intensity control, the value at each wavelength which would deliver equal amounts of energy could be calculated.

This method, together with the control of the voltage across the light source of the colorimeter, enabled the calibration to be repeated on successive days to within $\pm 2\%$.

(d) *Spectral energy content (absolute)*. The spectral light at wavelength $520\text{ m}\mu$ was matched visually by a flicker method with white light derived from a source of known candle power (Wright 1946).

(ii) *Determination of the Photopic Luminosity Function*

Two observers with normal colour vision were investigated. Before any series of measurements, one subject (L.C.T.) undertook a period of 5 min. dark adaptation, whereas the other (W.D.W.) began measurements immediately upon arrival at the colorimeter. The settings at the various wavelengths throughout the whole spectrum were made in random order, so that the effects of dark adaptation, if any, would not produce a systematic variation throughout the spectrum and would be included in the errors of the observations. Since these errors remained small at all wavelengths, it was concluded that dark adaptation did not affect the present measurements to any great extent.

Luminosities at a convenient number (about 20) of wavelength settings were measured at each observing session, and the randomization continued until three readings at each wavelength had been obtained. Each of these three readings was the mean of two settings of the intensity control of the colorimeter.

Owing to the low sensitivity of the eye to extreme red and blue wavelengths, it was not possible to measure the luminosity at all wavelengths at the highest luminance level because of the limit imposed by the energy delivery of the colorimeter. Accordingly, measurements were taken in four sections and the whole curve synthesized from its parts and presented as though it had been measured at one luminance level. Determinations were made every $10\text{ m}\mu$ from 400 to $700\text{ m}\mu$.

From 490 to $680\text{ m}\mu$ (L.C.T.) and from 480 to $670\text{ m}\mu$ (W.D.W.) at a luminance level of 7.62 candle/ft^2 measurements were made with a flicker method. The energy E_λ at each wavelength λ required to match the light from the standard white source could then be calculated.

From 500 to $460\text{ m}\mu$ (L.C.T.) and from 490 to $450\text{ m}\mu$ (W.D.W.) measurements were made with a similar flicker method but at a luminance level of 0.79 candle/ft^2 .

From 470 to $410\text{ m}\mu$ (L.C.T.) and from 460 to $410\text{ m}\mu$ (W.D.W.) the wavelength to be measured was compared directly in brightness with the comparison patch, which was of wavelength $440\text{ m}\mu$ and at a luminance level of 0.019 candle/ft^2 . The flicker method was abandoned at these wavelengths because of the difficulty of making an accurate comparison with the slow flicker speeds necessary.

From 670 to $700\text{ m}\mu$ (L.C.T.) and from 660 to $700\text{ m}\mu$ (W.D.W.) wavelength $670\text{ m}\mu$ at a luminance level of 1.74 candle/ft^2 was used as a comparison field.

The luminosity was expressed as $\log(1/E_\lambda)$. For observer L.C.T. the values for wavelengths 480 to $460\text{ m}\mu$ were joined to those for wavelengths 490 to $680\text{ m}\mu$ by determining from the readings at 500 to $490\text{ m}\mu$ the factor by which the energy of the comparison field had been reduced, and then decreasing the values of $1/E_\lambda$ obtained from wavelength 480 to $460\text{ m}\mu$ by this factor. Thus the whole curve is

presented as though it had been measured at the highest luminance. Each of the other sections was overlapped in a similar way. The maximum ordinate of the whole curve was then reduced to a value of 2.00.

In calculating the errors of the observations, a value for the logarithm of the luminosity was obtained from each colorimeter setting. Then the standard error of the mean was calculated according to the formula $\sqrt{[\Sigma(x - \bar{x})^2/n(n-1)]}$, where x is the logarithm of any measurement of luminosity, and $\bar{x}$ the mean of n such readings. Typical values for σ_m , the standard error of the mean, were: L.C.T., 520 $m\mu$, 0.012 log units; W.D.W., 640 $m\mu$, 0.009 log units.

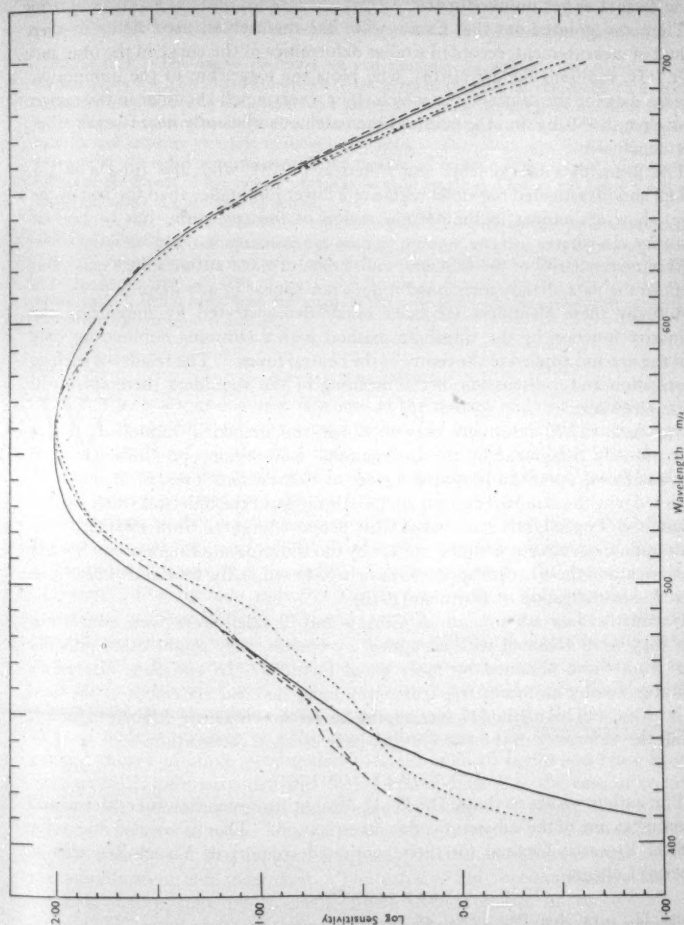
§ 3. RESULTS

Both observers had normal colour vision and the values for each, which are the means of three settings at each wavelength, are given in the Table and the Figure. Also in the Figure are shown the mean curve for Gibson and Tyndall's 26 'good' observers and the C.I.E. standard photopic luminosity curve, both plotted with

The Logarithms of the Ordinates of the Luminosity for the Two Observers, W.D.W. and L.C.T., given for each 10 $m\mu$ of Wavelength. The results are presented in four separate series as measured.

Wavelength ($m\mu$)	L.C.T.		W.D.W.	
	at 0.019 c/ft ²	at 0.79 c/ft ²	at 0.019 c/ft ²	at 0.79 c/ft ²
410	1.650		0.122	
420	0.067		0.389	
430	0.302		0.520	
440	0.414		0.679	
450	0.495		0.725	0.729
460	0.602	0.611	0.812	0.808 ^a
470	0.779	0.770	at 7.62 c/ft ²	0.995
480	at 7.62 c/ft ²	0.899	1.112	1.123
490	1.023	1.039	1.237	1.225
500	1.290	1.273	1.429	
510	1.526		1.621	
520	1.756		1.790	
530	1.866		1.881	
540	1.913		1.939	
550	1.972		1.988	
560	2.000		2.000	
570	1.951		1.965	
580	1.890		1.913	
590	1.822		1.815	
600	1.745		1.749	
610	1.663		1.659	
620	1.552		1.546	
630	1.377		1.355	
640	1.209		1.204	
650	1.032		0.982	at 1.74 c/ft ²
660	0.753	at 1.66 c/ft ²	0.747	0.733
670	0.465	0.462	0.418	0.432
680	0.159	0.162		0.114
690		1.837		1.743
700		1.461		1.373

logarithmic ordinates. Gibson and Tyndall's 'apparent irregularities' in the normal luminosity curve have been clearly confirmed, and the discrepancy between the mean curve for Gibson and Tyndall's observers and the C.I.E. standard curve is also shown.



A comparison of the photopic luminosity functions of two observers with normal colour vision, with Gibson and Tyndall mean luminosity function for 26 'good' observers, and with standard C.I.E. function.
 — C.I.E.; - - - Gibson and Tyndall's mean curve; ····· observer L.C.T.; - · - · - observer W.D.W.

§ 4. DISCUSSION

Blue irregularities. Use of the logarithm of the sensitivity as ordinate undoubtedly enhances the appearance of the shoulders. If the data are plotted with ordinates on the more usual percentage basis, the shoulders do indeed seem to be merely 'irregularities', and these findings serve to stress the advantage of the log plot when the 'tails' of the curve are under examination.

Gibson and Tyndall were, of course, aware of the shoulders, and could find nothing to account for them in their experimental technique. The individual curves for eight of their observers, taken at random from their paper, all show the deformities of the curve. Gibson and Tyndall pronounced the shoulders to be due to factors as yet uninvestigated.

They also pointed out that Exner, who, like themselves, used a step-by-step method of measurement, recorded similar deformities of the curve in the blue and violet. In addition, Bender (1914), who plots the logarithm of the luminosity, provides data for the central fovea which show a very small shoulder in the region of wavelength 480 $m\mu$, but the points chosen are not sufficiently near to each other to be conclusive.

The mean data of Coblentz and Emerson (1918), who also used a flicker method and investigated the violet region at a lower luminance than the rest of the curve, show a shoulder in the 440 $m\mu$ region of the spectrum, but no regular deformity elsewhere.

Much irregularity of the blue and violet data of most authors does exist, but except in the data already mentioned it does not appear in any definite form.

Actually these shoulders are more easily demonstrated by measuring the luminosity function by the threshold method with a stimulus subtending only 15' at the eye and applied to the centre of the central fovea. The results of such an investigation and a discussion of the meaning of the shoulders themselves will appear elsewhere.

Standard C.I.E. luminosity curve. Since the standard luminosity curve internationally recognized by the International Commission on Illumination in 1924 was based upon the proposed figures of Gibson and Tyndall, it might be wondered why the standard curve is smooth throughout the blue and violet region. Gibson and Tyndall, however, based their proposed figures from 490 $m\mu$ downwards upon those already adopted in 1918 by the Illuminating Engineering Society of America, and these in turn appear to have been based, in the blue and violet, upon the single investigation of Hartman (1918).

Hartman's data are smooth, as Gibson and Tyndall were well aware, but since they were obtained with an optical pyrometer, they might quite possibly differ from those obtained by more usual methods. In addition, Hartman's apparatus was by no means free from stray light, no clear statement of the field size is given, and his method of overlapping his sections is a little difficult to follow, so that the deformity in the curve might have escaped recognition.

ACKNOWLEDGMENTS

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The Acoustic Characteristics of Conical Pipes

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ABSTRACT. The paper describes an experimental investigation on slightly tapering conical pipes used as sound transmitters, undertaken in order to determine their acoustic characteristics, their end corrections and the pressure variation along the inside of the pipes.

A hot wire anemometer and a Pitot pressure tube were used for recording particle velocity and pressure variation in the acoustic field. The results obtained for the resonant frequencies show fair agreement with the theoretical formula $kr = -\tan kl$, where r is the radius of the throat of the conical pipe, l is its slant length and $k = \omega/c$, ω being the angular frequency and c the velocity of sound.

The recording of the acoustic pressure along the axis of the conical pipes shows that the nodes inside are places of minimum but not zero pressure, and that they are not equidistant. The end corrections are deduced from these pressure curves. The correction is always positive, being less than that of a cylindrical pipe of diameter equal to that of the mouth of the cone. It increases with the angle of the cone and with the order of the overtone to which the conical pipe resonates.

§1. INTRODUCTION

WHEN a sound source is placed at the narrow end of a conical pipe, the wave travels along the cone to the wide end, where it is in general partly radiated to the free space and partly reflected back into the cone. Under suitable conditions the cone may act as an efficient radiator, and the radiated energy be concentrated in a narrow beam.

According to Rayleigh, the radiation from a narrow cone obeys the diffraction law for a circular hole $\sin \theta = 0.61\lambda/R$, where θ is the plane semi-angle of divergence of the radiated beam, R is the radius of the mouth, and λ the wavelength of the radiation. The formula indicates that a cone with a wide mouth is a good directional radiator for high frequency sounds; but it does not relate the divergence to the angle or length of cone. Moreover, Rayleigh states that the pressure at a point far from the mouth of the cone is inversely proportional to its angle.

Early experimental work on conical horns has been done by Blaikley (1878), who used horns complete to the vertex and determined the positions of the internal nodes by means of a thin metal diaphragm. In 1920, Webster gave a general theory of wave propagation in horns of small angles and finite lengths. Stewart (1925) has extended this theory and applied it to the case of receiving horns. He tested his results experimentally using a Rayleigh disc, which he placed at the vertex of the horn to measure the intensity of the sound received. He has pointed out that for each frequency there is an optimum angle at which the amplification is a maximum. The value of this optimum angle has been deduced theoretically by Hoersch (1925). Working also on receiving cones, Foley (1922) used spark photography to trace the wave within the cone, and a Rayleigh disc to measure the intensity at the narrow end. The total of present information available is, however, inadequate to formulate the full characteristics of cones, especially when used as transmitters.

Research on end corrections of conical pipes has been limited and available results are controversial; for we find that Stewart (1925) uses the value 0.7R

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as a positive end correction for all conical pipes where R is radius of the mouth, while Bate and Wilson (1938) give the expression $(1.6 - 1.2\alpha)$ for the end correction of a cone of semi-apical angle α° , where 1.6 is the end correction for a cylindrical tube of radius 2.7 cm.

The object of the present investigation is to obtain experimental information on these points, and to compare the results with those deduced theoretically.

§ 2. METHOD

Experiments have been carried out on 14 conical pipes, each of zinc, 1.5 mm. thick but of varying dimensions. The cones were grouped in sets according to the factors to be studied, namely, angle, length, diameter of mouth, etc. The study included the fundamental and the first two overtones of each cone.

The cone under test is excited by a controllable calibrated sound source applied at the narrow end. The acoustic field is studied by two devices, a hot wire velocimeter or anemometer and a Pitot pressure tube. The former gives a measure of the mean velocity of vibrating air particles, and the latter gives a measure of the acoustic pressure within the cone.

Detailed description of the construction and the mode of action of the hot wire velocimeter is given by Richardson (1926), who was the first to use it for acoustic field measurements. This velocimeter makes use of the drop in resistance of an electrically heated platinum wire when subjected to the acoustic field. It has the advantage of causing a minimum of disturbance to the field.

The Pitot tube is an instrument originally used for measurement of pressure head in a steady fluid flow. It is a narrow tube with a fine hole at one end and communicating with a sensitive manometer at the other end, so that if placed facing a steady stream, a steady pressure is developed inside it which can be measured by a manometer. It is shown here that when this tube is placed along a progressive or a stationary wave it gives a measure of the acoustic pressure at the point where its end is situated. When compared with other similar devices, such as the Rayleigh disc, it is found to be simpler in operation and less disturbing to the field.

§ 3. APPARATUS AND PROCEDURE

The source of sound is a moving-coil loudspeaker unit actuated by a beat-frequency oscillator whose readings are calibrated against a standard oscillator. The purity of its tone is checked by means of cathode-ray oscillograms taken at different frequencies. The sound is applied to the cone through a narrow neck 2 cm. long and 1.5 cm. radius. The constancy of the acoustic output of the loudspeaker for all frequencies is maintained by means of a hot wire grid placed within the neck; the reading of this grid is kept constant throughout the experiment.

To study the disposition of the acoustic field within the conical pipe another hot wire anemometer is used, carried by a suitable traversing mechanism which can move to any position in the field. The cone under test and the measuring device are installed within an acoustically dead chamber lined with a layer of hair felt 2.5 cm. thick and another layer of cotton wool 10 cm. thick. The internal size of the chamber is about 2.5 cubic metres, and the top side of the chamber facing the mouth of the cone is removed.

The movable hot wire anemometer consists of a platinum wire 1 cm. long and 0.0025 cm. in diameter, soldered to the ends of two thin brass needles mounted on ebonite. These needles exert a slight tension on the wire to keep it taut.

The heating current is 160 ma., a value far below that required to heat the wire to dull red (280 ma.). The steady fall in the resistance of the wire when subjected to the sound field is measured by a sensitive Wheatstone bridge of which the hot wire forms one arm. The decrease in resistance of the hot wire when exposed to the sound field is very small, and is taken as proportional to the deflection in the circuit galvanometer. The E.M.F. of the source supplying the network is stabilized by a separate circuit.

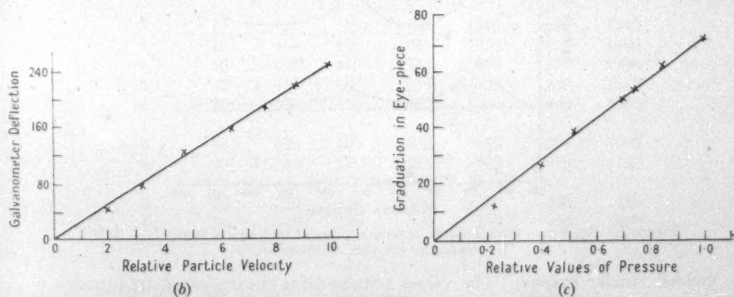
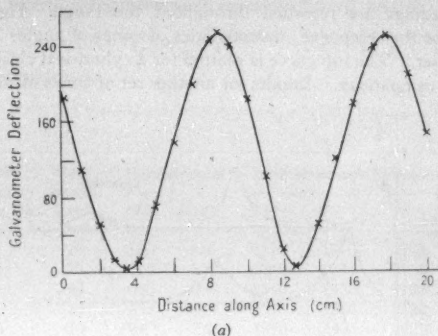


Figure 1.

- (a) Readings of galvanometer as the hot wire is moved along the axis of a tube.
 (b) Calibration curve for hot wire anemometer.
 (c) Calibration curve for Pitot pressure tube.

The hot wire anemometer is calibrated by moving the wire along the track of a stationary wave formed inside an open tube to which a sound source is applied near one end. The galvanometer deflections are plotted against the positions of the hot wire for overtones of the sound source (Figure 1(a)). Figure 1(b) gives the calibration curve.

The Pitot pressure tube consists of a uniform brass tube 15 cm. long, 0.15 cm. internal diameter and 0.5 cm. external diameter. It is terminated at one end by a hole 0.02 cm. in diameter; its other end is connected to a sensitive xylol manometer in which the position of the liquid meniscus is read against the graduated eyepiece of a microscope. The pressure tube is calibrated by a method similar to that for the hot wire anemometer; the calibration curve is given in Figure 1(c).

§4. RESULTS AND CONCLUSIONS

(i) *The Resonance Characteristics of Conical Pipes*

A set of cones is chosen, each of slant length 20 cm. and throat radius of 1.5 cm., but having semi-apical angles of 2, 4, 6, 8, and 10° respectively. The movable hot wire anemometer is placed at the centre of the mouth of the cone to be examined. The input frequency is then changed gradually over a range covering the fundamental and the first two overtones of the cone, and the anemometer readings are recorded throughout the range. Figure 2 shows typical curves for the resonance characteristics of cones of angles 2, 4, 6 and 8° from the above set. The top curve is plotted for a cylindrical closed pipe of the same length for comparison. Results for another set of cones of different length

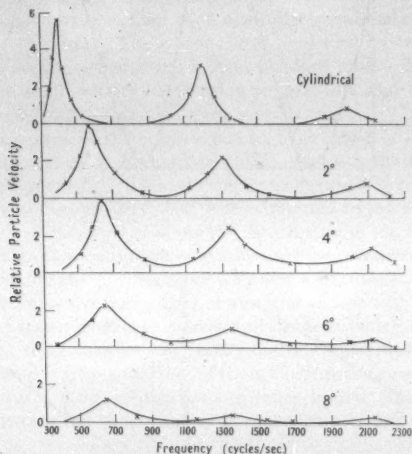


Figure 2. Resonance characteristics of conical pipes of semi-apical angles 0, 2, 4, 6, 8°, recorded by hot wire anemometer.

yielded similar graphs. The values obtained for the resonant frequencies of the cones used are given in the Table. These show that (a) the fundamental and overtone frequencies increase as the angle of the cone increases; (b) the overtones are inharmonic; (c) the output intensity at the mouth of the cone for the fundamental and overtones decreases as the angle increases.

In order to compare these results with those obtained theoretically, we shall derive an equation for resonance in truncated conical transmitting pipes in a way similar to that used by Stewart for his receiving cones.

The equations for the particle velocity v and the pressure p at a point along a stationary wave formed within a conical pipe are

$$v = -\left\{\frac{1}{r} + ik\right\}A/r\}e^{ik(ct-r)} - \left\{\frac{1}{r} - ik\right\}B/r\}e^{ik(ct+r)}, \quad \dots\dots (1)$$

$$p = -\{ik\rho c/r\}\{Ae^{ik(ct-r)} + Be^{ik(ct+r)}\}, \quad \dots\dots (2)$$

where r is the distance of the point from the apex of the cone, A and B are two constants depending on the boundary conditions, and ρ is the mean density of air. The two constants A and B are complex quantities and can therefore be replaced by $(a_1 + ia_2)$ and $(b_1 + ib_2)$ respectively.

Regarding the throat of the truncated conical pipe where the sound is applied as a closed end ($r=r_1$), we have

$$0 = -\{(1/r_1 + ik)(a_1 + ia_2)/r_1\}e^{ik(ct-r_1)} - \{(1/r_1 - ik)(b_1 + ib_2)/r_1\}e^{ik(ct+r_1)},$$

the real part of which is

$$0 = (a_1 + b_1)(\cos kr_1 + kr_1 \sin kr_1) + (a_2 - b_2)(\sin kr_1 - kr_1 \cos kr_1). \quad \dots\dots (3)$$

Similarly, the mouth is regarded as an open end ($r=r_2$), so that

$$0 = -\{ikpc/r_2\}\{(a_1 + ia_2)e^{ik(ct-r_2)} + (b_1 + ib_2)e^{ik(ct+r_2)}\},$$

the real part of which is

$$0 = (a_1 + b_1) \cos kr_2 + (a_2 - b_2) \sin kr_2. \quad \dots\dots (4)$$

Semi-apical angle ($^\circ$)	Slant length (cm.)	Fundamental		First overtone		Second overtone	
		exp.	theor.	exp.	theor.	exp.	theor.
2	20	570	530	1310	1310	2105	2130
4	20	620	620	1330	1370	2115	2190
6	20	630	660	1340	1420	2120	2220
8	20	635	670	1345	1450	2120	2240
10	20	640	720	1350	1470	2125	2280
2	30	410	390	900	880	1430	1440
4	30	440	430	925	950	1455	1480
6	30	450	465	945	960	1470	1490
8	30	450	480	955	980	1480	1515
10	30	450	500	960	1000	1500	1540
2	40	320	300	680	690	1094	1090
4	40	330	340	715	725	1170	1120
2	50	270	250	560	560	870	880
4	50	280	280	560	590	850	900

From equations (3) and (4) we get

$$\begin{aligned} (a_1 + b_1)/(a_2 - b_2) &= -\tan kr_2 \\ &= -\frac{\sin kr_1 - kr_1 \cos kr_1}{\cos kr_1 + kr_1 \sin kr_1}, \\ \tan kr_2 &= \frac{\cos \phi \sin kr_1 - \sin \phi \cos kr_1}{\cos \phi \sin kr_1 + \sin \phi \cos kr_1}, \end{aligned}$$

where $\tan \phi = kr_1$; $\tan kr_2 = \tan (kr_1 - \phi)$.

$$kr_2 - kr_1 = -\tan^{-1} kr_1; \quad kr_1 = -\tan^{-1} kl, \quad \dots\dots (5)$$

where l is the slant length of the cone.

This equation gives the resonant frequencies of the truncated cone when used as transmitter, the end correction being neglected. The equation is solved graphically, and the numerical values for cones examined are given, together with the experimental results, in the Table. The overtones are found to be inharmonic in the two results.

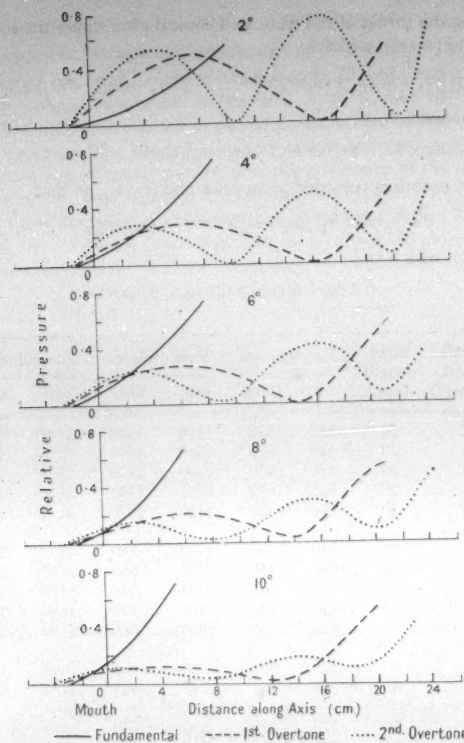


Figure 3. Pressure variation along the axes of cones recorded by Pitot pressure tube. (Figures indicate semi-angle.)

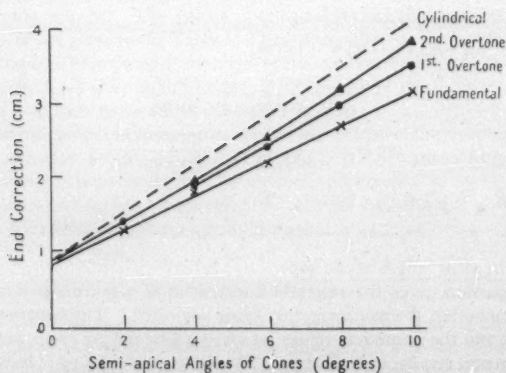


Figure 4. End corrections for conical pipes.

(ii) *The Pressure Variation along the Axis of the Conical Pipe*

The movable hot wire anemometer is then replaced by the Pitot pressure tube, which can also be moved along the axis of the cone. The readings for the acoustic pressure along the axis are recorded when the conical pipe resonates to its fundamental and also to its first and second overtones. Typical results for cones of slant length 30 cm. and semi-apical angles 2, 4, 6, 8 and 10° are given in Figure 3. The curves show that:

- (a) The amplitudes of the stationary waves within the conical pipe increase as we move inwards. This becomes clear as the angle of the pipe increases, or as the pipe resonates to a high overtone.
- (b) The pressure nodes formed within conical pipes are pseudo-nodes, i.e. places of minimum but not zero pressure. Near the throat these nodes tend to become true nodes, especially for cones of small angles.
- (c) When the conical pipe resonates to a high overtone, the pseudo-nodes will not be equidistant. They are nearer together towards the mouth than towards the throat, and are less clearly defined the larger the angle of the cone.

(iii) *End Corrections*

The end corrections for the different conical pipes are deduced from the curves of Figure 3 by measuring the intercept on the x axis as the curve is extrapolated.

The values of the end corrections for the set of cones of length 30 cm. are plotted in Figure 4 against the semi-apical angles of the cones for resonance to the fundamental, first overtone, and second overtone respectively. The broken curve gives the corresponding corrections for cylindrical pipes of radii equal to those of the mouths of the cones used. An examination of these curves reveals the following:

- (a) The end correction is always positive, and is less than that of a cylindrical pipe having the same radius as that of the mouth of the cone.
- (b) It increases as the tube resonates to a higher overtone, approaching monotonically that of a cylindrical pipe.
- (c) It decreases with decrease of angle of the cone, and follows approximately the empirical equation

$$C = 0.6a + (0.6l - 2100/n) \sin \theta,$$

where C is the end correction in cm., a is the radius of the throat, l is the slant length, n is the resonant frequency of the cone, including fundamental or any higher overtone, and θ is the semi-apical angle in degrees.

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Measurements of the Velocity of Sound in Liquid Argon and Liquid Methane

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ABSTRACT. The ultrasonic velocity in liquid argon and methane has been measured between their boiling and melting points and the ratio of specific heats calculated. The compressibilities and degrees of association have also been calculated and compared with the results obtained for other liquefied gases (hydrogen, helium).

§1. INTRODUCTION

CONTINUING our measurements on the velocity of ultrasonics in liquefied gases (van Itterbeek and de Bock 1949), we have now measured velocities in liquid argon and methane under their saturation pressures, using a source of frequency 523 kc/sec. in an acoustic interferometer. The range of temperature is from 87.25°K. to 83.75°K. in liquid argon and from 112°K. to 90.6°K. in liquid methane, these being the respective boiling and melting points. The latter gas was tested in a model of the cryostat previously described which was adapted to reach the desired temperatures. There are no previous data for methane, but Liepman (1939) used the optical diffraction method at 7.5 Mc/sec. in liquid argon containing 2% nitrogen. Galt (1948) has recently used the pulse technique at 44.4 Mc/sec. in argon near its melting point. The argon used by us was supplied by the Philips Lamp Works of Eindhoven and as it had a purity of within 0.1% our results are about 1% higher than the velocities of Liepman.

§2. EXPERIMENTAL METHOD AND APPARATUS

Figure 1 shows the interferometer, which is of the same type as that used for our measurements on gases at low temperatures except that we have modified the quartz mounting, which is now as shown in Figure 2. It is so arranged that the pressure of the upper electrode on the crystal can be changed at will by means of springs, *d* (*a* is a massive brass plate). The crystal is connected in parallel with the condenser of a tank circuit tuned to an oscillator which is driven by a Philips B424 one-watt valve. The changes of impedance as the length of the vibrating column passes through an integral number of half wavelengths are detected by the voltmeter valve (Philips AF7) and registered on the galvanometer.

§3. MEASUREMENTS IN LIQUID ARGON

The method of operation follows from Figure 3. *A* represents the high-pressure cylinder containing argon gas. The cryostat *D* surrounding the resonator tube (*B*, 4 cm. diam.) is filled with liquid oxygen. By this means temperatures between 87.25 and 83.75°K. are realized. The argon gas is condensed into *B* to a height of 8 cm. above the quartz. Before each measurement

the argon is solidified in the apparatus and pumped off by means of a Leybold rotary pump and during the measurements the pressure above the liquid argon

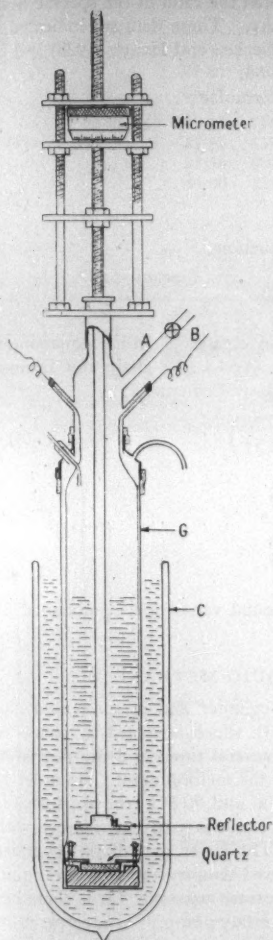


Figure 1.

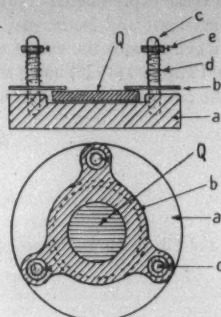


Figure 2.

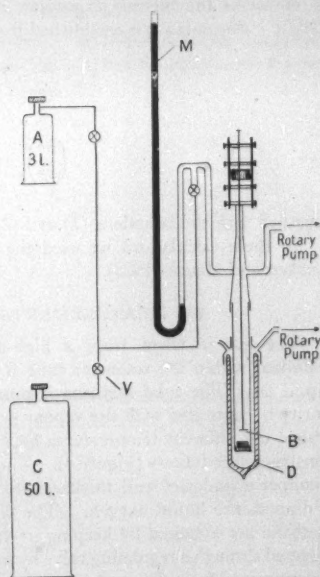


Figure 3.

is kept constant at its saturation pressure by letting it escape past the regulator valve V into the cylinder C, of capacity 50 litres and previously evacuated. A test of the frequency of the source was made at each temperature on a wavemeter

(accurate to within ± 0.01 kc/sec.) and five checks on the velocity were made, covering ten half wavelengths. Results are given in Table 1.

From these experimental results we calculated the ratio of the specific heats and the adiabatic and isothermal compressibility. These data are indicated in Table 2 together with the values obtained by Eucken and Hauck (1928) and by Simon and Kippert (1928) using the static method.

For the calculations we used the following formulae:

$$\beta_{is} = \frac{C_p}{C_v} \beta_{ad} \quad \dots\dots (1)$$

$$\beta_{ad} = \frac{1}{\rho V^2} \quad \dots\dots (2)$$

The value of C_p/C_v can be obtained from the equation:

$$\frac{C_p}{C_v} - 1 = \frac{T\alpha^2 V^2}{C_p} \quad \dots\dots (3)$$

C_p being the specific heat at constant pressure in ergs/gm. In the equation (3) α represents the thermal expansion coefficient (4.54×10^{-3} (Baly and Donnan 1902)). Formula (3) is established from the classical equations:

$$C_p - C_v = T \left(\frac{\partial p}{\partial T} \right)_v \left(\frac{\partial v}{\partial T} \right)_p \quad \dots\dots (4)$$

$$\left(\frac{\partial v}{\partial p} \right)_T = - \left(\frac{\partial T}{\partial p} \right)_v \left(\frac{\partial p}{\partial T} \right)_v \quad \dots\dots (5)$$

together with the equations (1) and (2).

In these calculations we used the experimental values for C_p obtained by Eucken and Hauck (1928).

§4. MEASUREMENTS IN LIQUID METHANE

The gas is taken from a high-pressure cylinder and is condensed and solidified within the resonator tube B (Figure 3), which is cooled by means of liquid air. The solid methane is pumped off several times and the degree of purity is controlled with the vapour pressure at the melting point. In order to realize the different temperatures between 112° K. and 90.6° K. the apparatus is constructed as follows (Figure 4). The resonator tube is partly surrounded with a copper cylinder of wall thickness 0.15 mm. The lower part of this cylinder is dipped into liquid oxygen. The different fixed temperatures for the liquid methane are obtained by keeping its vapour pressure constant. It is therefore pumped through a regulating valve by means of a rotary pump. The temperature is determined from the vapour pressure (read on the manometer M), by using the vapour pressure curve measured by Stock, Henning and Kuss (1921). In a recent publication (van Itterbeek and de Bock 1949) on the velocity of sound in liquid nitrogen, we mentioned a few measurements in liquid methane at the melting point. We found 1,510 m/sec. with an error of about 10%. At that time the measurements in liquid methane were very difficult because of the secondary maxima. For the present measurements we increased the tension on the upper electrode of the quartz crystal. The new measurements are very regular and

Table 1

Date	T (°K.)	p (mm.)	V (m/sec.)	V (m/sec.) Liepman	V (m/sec.) Galt
28.2.49	87.35	767	849	835	853 (87.2° K.)
	85.80	645	857	848	
	83.95	525	875	865	
11.3.49	87.15	754	845		
	85.85	653	860		
	85.30	610	868		
	84.00	529	780		

Table 2

T (° K.)	V (m/sec.)	$\frac{C_p}{C_v} - 1$	p^* (gm/cm ²)	β_{ad} exper.	β_{is} exper.	β_{is} (a)	β_{is} (b)
87	850	1.17	1.403	99	218	245	180
86	858	1.18	1.409	97	212		
85	866	1.19	1.416	94	205		
84	874	1.20	1.423	92	200		

β_{ad} = adiabatic compressibility in 10^{-12} cm²/dyne. (a) Results of Eucken and Hauck.

β_{is} = isothermal compressibility in 10^{-12} cm²/dyne. (b) Results of Simon and Kippert.

* Baly and Donnan

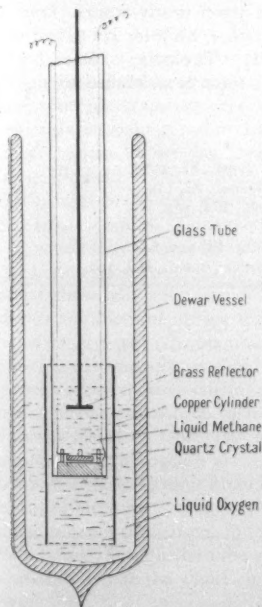


Figure 4.

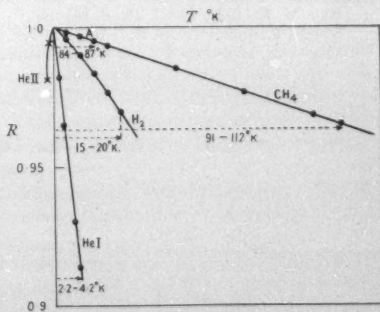


Figure 5.

accuracy to within 1% was obtained. The experimental values are given in Table 3.

Table 3
Velocity of Sound in Liquid Methane

Date	T (° K.)	p (mm.)	V (m/sec.)
25.3.49	111.4	746	1418
	108.8	602	1461
	102.6	332	1485
	94.9	148	1545
28.3.49	111.7	764	1414
	105.6	458	1462

§5. CONCLUSIONS

From our measurements on argon and methane we calculated the expression V^3/ρ given by Rao (1940) and also what is called by him the relative association: $R = \{V_m^3/\rho_m\}/\{V^3/\rho\}$, V_m and ρ_m corresponding with the melting point or another characteristic temperature of the liquid. So for example in the case of helium we took the lambda point.

The expression of Rao has no thermodynamical meaning; it is purely empirical, but it gives an indication of the degree of association relative to the melting point.

It is remarkable that for He I, methane and argon nearly straight lines are obtained for R as a function of the absolute temperature, while for He II, hydrogen and other liquids curves are obtained (see Figure 5). The values of R for the case of hydrogen are calculated from measurements recently published by us (van Itterbeek and Verhaegen 1949).

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The Behaviour of the Cathode Spot on an Undisturbed Mercury Surface

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ABSTRACT. The cathode spot of transient mercury arcs has been studied by means of a Kerr cell apparatus capable of taking many superimposed photographs. Discussion is limited to times when the spot is moving with sufficient velocity not to disturb the mercury surface. It is found that for a given current the "spot" takes the form of a line, or broken line, of total length proportional to the current. This is also true for a spot in a magnetic field. If the current rises slowly, the length of the line increases proportionally to the current; but if the rate of growth of current is greater than a value lying between $\pi \times 10^7$ and $2\pi \times 10^7$ amperes per second, then fresh spots form and spread out radially from the point of formation into lines moving with a maximum velocity of about 10^4 cm/sec. It is found that this velocity is never greatly exceeded whatever the experimental conditions. The current density is in excess of 10^8 amperes/cm².

§ 1. INTRODUCTION

By means of single Kerr cell photographs of exposures of one microsecond or less it has been shown that the current density of the emitting cathode spot for transient mercury and other arcs is much higher than was previously thought (Froome 1948 a). Cobine and Gallagher (1948) have also deduced values higher than those obtained hitherto for cathode spots moving in a magnetic field. Their values were arrived at by measuring the duration of the flash of light as the spot moved under a slit and from markings on the surface of some solid cathodes. Their method did not resolve the actual shape of the emitting area, which they assumed to be circular.

Single exposures of the cathode spot suffer from the disadvantage that they cannot show how any individual "spot" behaves or moves under the action of various conditions of current and rate of growth of current. The results described in this paper have been obtained by means of a Kerr cell apparatus which can give a sequence of exposures varying from one-tenth to six microseconds. The interval between each exposure can be varied from 0.4 to 70 microseconds, and the number in a sequence can also be varied. This apparatus has been described elsewhere (Froome 1948 b). Observations are made by means of low-power photomicrographs using the optical arrangement shown in Figure 1. It is thus possible to observe the behaviour of an individual spot for times of more than 100 microseconds if necessary. The transient arcs are obtained by discharging a condenser, or a number of condensers, in the form of an "artificial line" charged to a hundred volts or so, through the tube and series resistance. The term "cathode spot" is used to denote the whole emitting area on the cathode irrespective of its exact form.

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The results to be discussed fall into two classes: those obtained at constant current, and those obtained with currents varying at a specific rate. In all the Kerr cell photographs reproduced the cathode spot starts from the bottom of the picture.

§ 2. RESULTS WITH CONSTANT CURRENT PULSES

Constant current pulses are obtained by discharging one of three "artificial lines" through the discharge tube. Each of these is composed of fifteen condensers and inductances, having a total capacity of $29 \mu\text{F.}$ and charged to various potentials up to 240 volts. Their characteristic impedances are 2 ohms, 0.5 ohm and

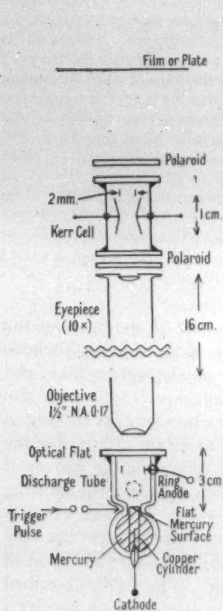


Figure 1.

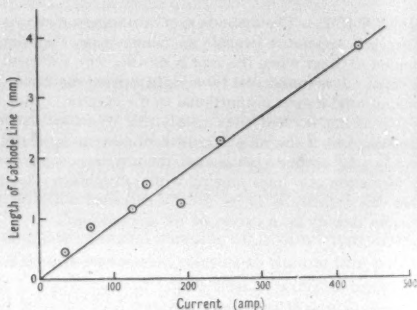


Figure 3.

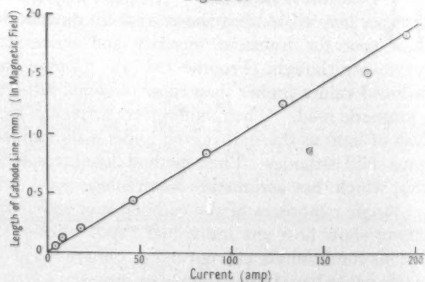


Figure 5.

0.2 ohm, giving constant current pulses of duration about $160 \mu\text{sec.}$, $40 \mu\text{sec.}$ and $15 \mu\text{sec.}$ respectively. The current obtained with each at the maximum voltage is 50 amp., 200 amp. and 450 amp., but intermediate currents are used by altering the initial applied voltages.

The discharge tube is triggered by means of a high tension pulse applied outside the tube adjacent to the cathode (Figure 1). The current in the tube is observed by means of a high speed oscilloscope. The pressure of the residual gas (air) in the discharge tube is about 0.01 mm. Considerable changes in the pressure do not produce any significant change in the behaviour of the cathode spot. The mercury cathode of the discharge tube is about 5 mm. in diameter.

This is made flat by wrapping a thin sheet of mercury amalgamated copper round the inside of the neck, the mercury surface being flush with the top of this sheet (Figure 1).

Figures 2(a), (b) (Plate I) show two multiple exposures of this cathode for constant current pulses of 145 and 190 amp. respectively. The exposure was $0.3 \mu\text{sec.}$ in each case, and the interval between exposures was $4.2 \mu\text{sec.}$ The cathode spot has started from the mercury surface adjacent to the glass wall opposite the triggering pulse, and fanned out in the form shown. Single exposures (not given here) show that the cathode spot takes the form of a broken line, or curve moving outward from the point of initiation. Figure 2 thus shows sequences of positions of this emitting area as it moves outward from the point of initiation at the bottom of the picture.

The velocity of the emitting line, which initially approaches 10^4 cm/sec. , becomes less towards the end of the discharge. When this velocity falls to the region of 10^3 cm/sec. the motion of the line becomes random and markedly slower. Each section of line then appears to cause an indentation in the mercury surface. The total length of the line is roughly constant during a discharge. This constancy has been observed for all such discharges with numerous different currents.

Figure 3 shows this in diagrammatic form as a graph of the total length of the emitting line against current. Each of the points represents the mean of the length of a number of such cathode lines at that particular current. It is seen that up to the maximum current here used (430 amp.) the total length of the emitting line is given by

$$l = Ki \text{ cm.}$$

where $K = 9.0 \times 10^{-4} \text{ cm. amp}^{-1}$. Over two hundred measurements were made in all, the maximum deviation being about 50% to either side of the mean value.

It was found that the initial outward velocity of the cathode spot from the wall of the discharge tube was never much in excess of 10^4 cm/sec. This point will be discussed later.

§ 3. CONSTANT CURRENT PULSES IN A MAGNETIC FIELD

It is well known that, in a magnetic field, the cathode spot moves retrogressively, with velocity increasing with field up to a maximum velocity of about 10^4 cm/sec. This has been studied by Smith (1942).

Figure 4(a) (Plate I) shows Kerr cell photographs of a 'spot' at 46 amp. moving in a magnetic field of 3,000 gauss. The exposures and intervals for Figure 4 are the same as for Figure 2. Again the emitting area takes the form of a broken line extending roughly perpendicular to the direction of motion. When the field is high there is less tendency for the line to break up.

Figure 4(b) shows a spot carrying a current of 60 amp. in a lower magnetic field (about 1,700 gauss). Again the total length of the emitting line appears proportional to the current. This photograph is interesting in that, soon after the start of the arc, a branch developed to the right of the main line; this branch gradually died out and the length of the remainder increased, thus maintaining the constancy of the total emitting length. At this lower magnetic field a tendency can be seen for the line to divide into smaller pieces, as is the case when no magnetic field is present.

Figure 4(c) shows a spot carrying a current of 10 amp. again in a field of about 1,700 gauss. Once more a branch can be seen. This branching is very common, especially at lower magnetic fields.

The total length of the emitting line is not affected by the strength of the field; this affects the velocity only.

Figure 5 shows the result of many measurements on similar photographs in the form of a graph of the length of the line l against current i . It is again seen that $l = Ki$ cm., where $K = 9.5 \times 10^{-4}$ cm. amp $^{-1}$. Within the limits of experimental error the two values of K are the same.

The width of the emitting line, allowing for movement during exposures, appears to be about 10^{-3} cm. Again no velocities much in excess of 10^4 cm/sec. were recorded.

§4. RESULTS WITH VARYING CURRENTS

The behaviour during current decay is straightforward. Whether the cathode spot is in a magnetic field or not, the total length of the emitting line decreases proportionally with decreasing current. If the spot is not in a magnetic field and the discharge is prolonged (lasting longer than, say, 20–30 μ sec.), unit spot formation (Froome 1948a) takes place. These spots then diminish in number with falling current.

From the results described, obtained with constant current pulses, it was to be expected that a rapidly rising current might have an interesting effect upon the cathode spot. As has already been mentioned, the velocity of the spot was never found to be greatly in excess of 10^4 cm/sec., yet the spot nearly always starts from a point on the mercury surface adjacent to the glass wall of the tube. The emitting area then takes the form of a line extending perpendicular to a radius from the starting point, and moving radially outward with a maximum permitted velocity of 10^4 cm/sec. (see Figure 7(a), Plate II). Hence its maximum length L after t sec. will be given (assuming the radius of curvature of the glass wall to be relatively large) by $L = 10^4 \pi t$. But the length is also related to the current i by the relation $L = Ki$ cm., where $K \approx 10^{-3}$. Hence $i = 10^7 \pi t$ amp. and $di/dt = 10^7 \pi$ amp/sec. represents the maximum value of rate of growth of current at which the emitting area can maintain itself in the normal form of a line moving with a maximum velocity of 10^4 cm/sec. If the cathode spot forms on the mercury surface away from the tube wall, the emitting line can move out radially in all directions, and the maximum rate of current rise will be twice the above value.

Figure 6, curve (a), shows the characteristics of a discharge with maximum rate of growth of current less than $10^7 \pi$ amp/sec. This was obtained by discharging a 4 μ F. condenser, initially at 240 volts, through an inductance of 75 μ H. and a resistance of 1.75 ohms in series with the discharge tube, the cathode spot being in a magnetic field of 1,700 gauss.

Figure 7(a) (Plate II) shows the behaviour of a cathode spot in the magnetic field during the whole discharge. The exposure is 0.3 μ sec. and the interval between each exposure is 4.2 μ sec. It is seen that the length of the emitting line (starting from the bottom of the picture) increases and then decreases, varying roughly proportionally with the current.

Figure 7(b) shows a cathode spot not in a magnetic field, the discharge characteristics being shown in Figure 6, curve (b). The potential drop across the tube was substantially constant at 30 volts for the greater part of the discharge time. Once



[Figure 4 (a).



Figure 4 (b).



Figure 4 (c).

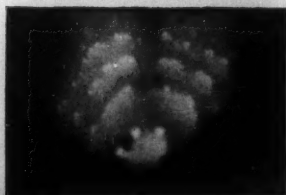
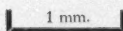


Figure 2 (a).



Figure 2 (b).

PLATE I.

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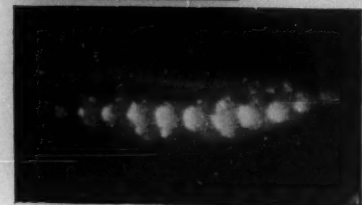


Figure 7(a).

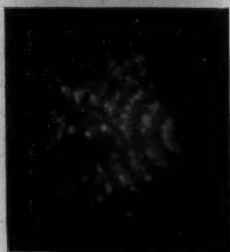


Figure 7(b).



Figure 9(a).

Figure 9(b).



Figure 10(a).

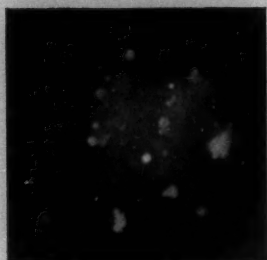
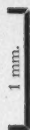


Figure 12.

Figure 10(b).

PLATE II.



more the increase and subsequent decrease in the length of the line is visible, corresponding to current rise and fall. The last four or five exposures do not show up as separate positions because of the small velocity of the spot. These results are to be expected from the studies with constant current pulses.

In the experiments with maximum rate of growth of current in the vicinity of 5×10^7 amp/sec. no magnetic field was used, in order that the behaviour should be as normal and as simple as possible. Figure 8 shows the characteristics

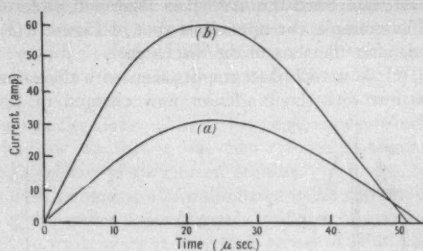


Figure 6.

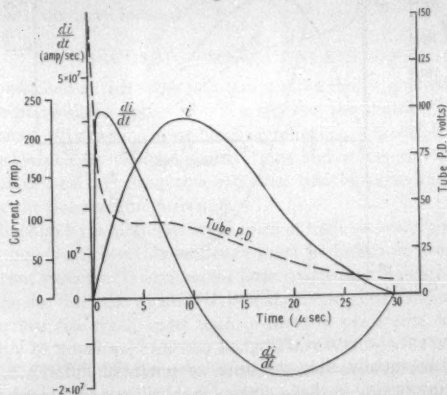


Figure 8.

observed by discharging a $20 \mu\text{F.}$ condenser, charged to 144 volts, through the tube and an inductance of about $2.5 \mu\text{H.}$ in series with a resistance of 0.2 ohm.

Figure 9(a) (Plate II) shows a sequence of exposures of a corresponding discharge with the condenser charged to 144 volts. It is seen that the emitting line is occupying all the length available for it, and that its velocity is found to be approximately constant at 10^4 cm/sec. instead of decreasing with time after the first microsecond or so. The exposure in each case is $0.1 \mu\text{sec.}$ and the interval between each is $1 \mu\text{sec.}$, and about the first five microseconds are shown. (In all photographs the spot—and hence time—starts from the bottom.)

Figure 9(b) shows another result taken under exactly the same conditions as Figure 9(a). Here, a number of spots have formed during the discharge. A particularly good example is seen at the top of the picture. Each of these new spots

forms the starting point of a more or less normal discharge sequence (i.e. similar to Figure 9(a)) which may, however, have a shorter life than the remainder of the discharge time. When these new areas have formed and commenced to spread out into circular lines, it is found that their maximum outward radial velocity is never greatly in excess of 10^4 cm/sec. Single exposures taken at a given time of similar discharges show that it is always the outermost emitting areas which are then carrying the current. Hence, by knowing the number of exposures given, it is possible to estimate when the new spots form and often also to estimate their life-time. For example, the uppermost spot of Figure 9(b) formed about three microseconds after the start of the discharge.

Figures 10(a), (b) show two photographs taken with the same exposure and circuit conditions but with the condenser now charged to 180 volts. This

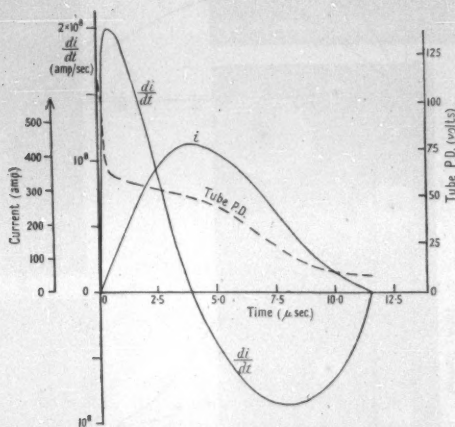


Figure 11.

increases the current and rate of change of current by a factor of about 1.3 times, but does not substantially alter the time, or potential difference, distribution. Figure 10(a) is interesting in that a single large brilliant spot can be seen to have formed probably between 3 and 4 μ sec. from the start, when the current was 200 amperes. At this time the only other emitting area was a short line of length corresponding to about 50 amp., so that for a very short space of time (under a microsecond) this large spot probably carried a current of 150 amp. If the whole of the bright area carried the current, the emission current density would be 1.5×10^6 amp/cm², but probably a part of the bright area shown is afterglow (see § 5) so that the current density was probably higher. The total life of this spot was under a microsecond, then two fresh spots formed and disintegrated radially. Both Figures 10(a) and (b) show rather more fresh spots to have formed than does Figure 9(b).

Hence we see that for high values of di/dt the cathode emitting area maintains its necessary length by forming fresh spots in front of the previous ones, and that these fresh ones disintegrate radially with the velocity mentioned.

The effect is even more pronounced at very high values of di/dt . Figure 11 shows the characteristics of the very rapid discharge obtained by discharging 20 μF . initially at 120 volts through the tube and a total inductance of about 0.3 μH . and negligible resistance. Figure 12 (Plate II) shows the cathode spot for about the first 3 μsec . of the discharge. The exposure is 0.1 μsec . and the interval is 1 μsec . This time quite a number of fresh spots have formed in front of the previously emitting areas during the interval between each exposure. These have presumably moved apart radially as before, but all the inner ones appear to have a life of under a microsecond. Even under these extreme conditions the emitting areas never move with a velocity much in excess of 10^4 cm/sec., but by virtue of the fact that fresh spots are constantly forming in front of the previous ones, the emitting area appears to move more rapidly than would be expected if the spots travelled at this velocity alone. The earlier single exposure technique would not reveal this behaviour, so that it was then erroneously thought that the spot actually reached a velocity of 10^5 cm/sec. or more.

The whole flash photograph of a discharge under extreme conditions, shown in Figure 8(a) of the earlier paper (Froome 1948 a), is an excellent example of the formation of these fresh spots at very high values of di/dt . For this discharge a peak value of rather more than 10^9 amp/sec. was recorded, and a very large number of new spots have formed.

§ 5. DISCUSSION AND CONCLUSIONS

It is not proposed at this stage to attempt more than a qualitative explanation of some of the phenomena observed. The fact that the emitting line never travels much faster than 10^4 cm/sec. must be of significance. Probably this represents the velocity at which neutralized positive ions and atoms of vapour are ejected from the spot in one direction at a rate that just balances the fresh numbers brought in by the positive ion current.

The mechanism by which new spots form at high values of the rate of growth of current is not clear, but this tendency must be linked in some way with the emission current density. There seems little doubt that the current density does not vary much, nor does the width of a normal emitting line; in fact, at high values of rate of current rise fresh spots tend to form in preference to an increase in current density of a normal emitting line. The emitting area is unstable unless in the form of a line, and when the current rise is too fast for the maintenance of a line, the freshly formed emitting spots have extremely short lives before they either disintegrate radially into normal lines or extinguish altogether.

The appearance of the cathode spot shown in all the photographs is most probably the appearance of a region of dense and highly excited vapour lying very close to the cathode surface.

C. G. Smith (1946) has photographed such a region for a cathode spot moving in a magnetic field by means of a microscope objective placed parallel to the mercury surface. He finds that for 10^{-3} cm. out from the mercury surface there is a suggestion of a "cathode dark space" emitting a continuous spectrum, and then for a further 4×10^{-3} cm. there is a very much more intense region of excited vapour.

The photographs shown here must pertain to this region. If this is so, the estimate of current density becomes difficult, for the width of the emitting line appears to be about 10^{-3} cm. (leading to an apparent current density of just over

10^6 amp/cm², but in reality this area must be at least 10^{-3} cm. out from the cathode. Hence the real width of the emitting line at the mercury surface might very well be smaller than 10^{-3} cm. It is, however, a reasonable assumption that a spot is symmetrical at the minimum current it will carry, and this gives us an alternative method for estimating the current density. If we assume that the width of the emitting line is equal to its length at its minimum current, we can estimate the current density more accurately. Experiments show that a cathode spot appears to extinguish at currents between $\frac{1}{2}$ and $\frac{1}{10}$ amp. The length of the spot would then be $\frac{1}{2}$ to $\frac{1}{10} \times 10^{-3}$ cm., and the current density might thus be roughly between 2×10^6 and 10^7 amp/cm². This value probably represents the best estimate obtainable by optical methods.

The present paper has been restricted to cases where the spot has been moving with a velocity sufficiently great not to disturb the mercury surface. This enables the spot to be observed easily. When the spot slows down to a velocity of about 10^3 cm/sec. it appears to create a tiny dent for itself in the mercury surface and its actual size becomes indefinite, probably owing to reflections from the sides of the dent, and the tendency for excited vapour to accumulate in it. This phenomenon constitutes the "unit spot" formation already mentioned.

It is thus the opinion of the author that when the "unit spot" and dent formation of prolonged discharges takes place, such spots are actually pieces of a broken-up line such that the total length would be given by $l \approx 10^{-3} i$ cm. The current density might still have the value given above.

The brightness of the cathode spot is greatest when it takes the form of a continuous line. This can be seen from the fact that photographs taken in a magnetic field strong enough to hold the line together are denser than those towards the end of a normal discharge when the cathode line has had time to start to break up into unit spots. This result would be expected, since the density of mercury vapour ejected from the cathode spot would be greater when it was in as confined an area as possible.

In all photographs there is seen what appears to be an afterglow, lasting a few microseconds, in the wake of the emitting area. This would presumably be caused by recombination in some of the more slowly moving ionized and excited vapour left behind by the spot.

ACKNOWLEDGMENTS

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Electron Flow in Curved Paths under Space-Charge Conditions

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ABSTRACT. Further developments in a method of obtaining solutions of space-charge flow, described in a previous paper, are presented. A single example of three-dimensional flow is given, as well as a number of general theorems. A distinction is drawn between "normal" flow, in which all electrons have the same energy (example : a thermionic device with a unipotential cathode) and "abnormal" flow, in which they may have different energies. It is shown that in the case of the former the application of the general method simplifies considerably.

§ 1. INTRODUCTION

IN a recent paper (Meltzer 1949, to be referred to as I) a method was described for obtaining rigorous solutions of electron flow subject to space-charge forces, and two examples were given. In the present paper another such example is given which is of interest because it is thought to be the first published solution of devious three-dimensional flow, i.e. a flow in which the trajectories are neither rectilinear nor parallel to a plane. There are also given some general results and theorems which should be of use in solving further problems.

§ 2. EXAMPLE OF THREE-DIMENSIONAL FLOW

Following the notation of the earlier paper, denote the unit vectors of the x , y and z axes of a right-handed rectangular cartesian system of axes by $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$.

Then a possible space-charge motion is one in which the electron velocity $\mathbf{v}$ at any point is given by

$$\mathbf{v} = \mathbf{I}(ax) + \mathbf{J}(by) - \mathbf{K}(a+b)z, \quad \dots\dots(1)$$

where a, b are any constants.

The charge density ρ , the electric field $\mathbf{E}$, potential V and current density $\mathbf{i}$ are given, respectively, by the following expressions:

$$\rho = (-m\epsilon_0/e)[a^2x^2 + b^2y^2 + (a+b)^2z^2], \quad \dots\dots(2)$$

$$\mathbf{E} = (-m/e)[\mathbf{I}a^2x + \mathbf{J}b^2y + \mathbf{K}(a+b)^2z], \quad \dots\dots(3)$$

$$V = (m/2e)[a^2x^2 + b^2y^2 + (a+b)^2z^2] + \text{const.}, \quad \dots\dots(4)$$

$$\mathbf{i} = (-m\epsilon_0/e)\{a^2x^2 + b^2y^2 + (a+b)^2z^2\}[\mathbf{I}ax + \mathbf{J}by - \mathbf{K}(a+b)z], \quad \dots\dots(5)$$

where e, m, ϵ_0 are respectively the electronic charge, the electronic mass and the permittivity of free space in rationalized M.K.S. units. The physical dimension of the quantities a and b is $(\text{time})^{-1}$.

It may be deduced from equation (1) that the trajectory equations are

$$x = Ae^{at}, \quad y = Be^{bt}, \quad z = Ce^{-(a+b)t}, \quad \dots\dots(6)$$

where A, B and C are constants with the dimension of distance and t is the time.

Hence each trajectory is given by the intersection of two surfaces:

$$xyz = \text{constant}, \quad x^b = (\text{constant})y^a. \quad \dots\dots(7)$$

Some of the physical characteristics of this flow can be immediately deduced from the above results: (i) the charge density is maintained at the same fixed value at every point of the space; (ii) the equipotentials are concentric similar ellipsoids with centre at the origin, but they cannot degenerate into spheres; (iii) each electron path lies entirely in the surface of a rectangular hyperboloid $xyz = \text{constant}$, and its trace on, say, the xy plane is a curve of the type $y = Kx^a$.

Since n may be odd or even, positive or negative, this trace may be a straight line through the origin, a curve of parabolic or cubic type through the origin, or of hyperbolic type with the x and y axes as asymptotes.

In principle this type of motion may be achieved by having an ellipsoidal conductor, parts of which emit electrons into the interior with the correct speeds and directions. In practice this would be extraordinarily difficult to achieve.

It may be noted that the first example of two-dimensional flow studied in I is a special case of the above, obtained by putting $b = -a$, which makes the ellipsoids degenerate into infinitely long coaxial circular cylinders.

§ 3. GENERAL RESULTS AND THEOREMS

In the previous paper it was shown that if the electron velocity vector $\mathbf{v}$ has components v_x , v_y and v_z and the acceleration vector $\mathbf{A}$ is formed from it as follows:

$$\mathbf{A} = \mathbf{I}(\mathbf{v} \text{ grad } v_x) + \mathbf{J}(\mathbf{v} \text{ grad } v_y) + \mathbf{K}(\mathbf{v} \text{ grad } v_z), \quad \dots\dots(8)$$

then the criterion for a possible space-charge flow is that the following two conditions should be satisfied:

$$\text{curl } \mathbf{A} = 0, \quad \dots\dots(9)$$

$$\text{div}(\mathbf{v} \text{ div } \mathbf{A}) = 0. \quad \dots\dots(10)$$

The direct application of (10) is sometimes cumbersome, and the criterion established by the following theorem may be more suitable.

Theorem 1

$\mathbf{v}$ will define a possible space-charge motion, provided the following two conditions are satisfied:

$$\text{curl } \mathbf{A} = 0, \quad \dots\dots(11)$$

$$\mathbf{v} \nabla^2 \mathbf{A} + \text{div } \mathbf{v} \text{ div } \mathbf{A} = 0, \quad \dots\dots(12)$$

where ∇^2 is the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

Proof. From equation (10) $\text{div } \mathbf{v} \text{ div } \mathbf{A} + \mathbf{v} \text{ grad div } \mathbf{A} = 0$. Now, in cartesian coordinates, $\text{grad div } \mathbf{A} = \text{curl curl } \mathbf{A} + \nabla^2 \mathbf{A}$, but from equation (9) $\text{curl } \mathbf{A} = 0$. Hence equation (12) follows.

Normal and Abnormal Flow

We proceed now to draw a distinction between two types of flow, one characteristic of most modern electronic devices, the other not so, but yet quite capable of realization. The historical reason for the predominance of the former is that it is associated with a single unipotential emitting cathode.

Writing η for the ratio e/m , we have, from Newton's second law, for an electron at any point,

$$\dot{\mathbf{v}} = \eta \text{ grad } V. \quad \dots\dots(13)$$

Multiplying both sides by $2\mathbf{v}$, one obtains $(d/dt)\mathbf{v}^2 = 2\eta\mathbf{v} \text{ grad } V$.

Now if $d\mathbf{s}$ represents the displacement of the electron along its trajectory in time dt at the point concerned, it follows that

$$d(\mathbf{v}^2) = 2\eta \, d\mathbf{s} \text{ grad } V = 2\eta dV, \quad \dots\dots(14)$$

where dV represents the corresponding change in potential along the trajectory. Hence

$$\mathbf{v}^2 = 2\eta V + K, \quad \dots\dots(15)$$

where K is a constant for the particular trajectory considered.

Equation (15) is, of course, nothing other than an expression of the conservation of energy of a single electron.

Two cases may arise: the normal case for which K is the same for every trajectory, and the abnormal case for which K is not the same for every trajectory.

In the normal case the speeds of electrons are the same everywhere on the same equipotential. This, for example, is the case in an ordinary valve with an indirectly heated cathode, where all the electrons leave the zero-potential cathode with zero speeds (always neglecting the Maxwellian thermal velocities); the flow described in §2 is also a normal one. An example of abnormal flow is that in a battery valve with directly-heated filament; owing to the voltage drop along the cathode, electrons will have the same speed on different equipotentials. The second example in I, viz. $\mathbf{v} = \mathbf{I}(ay) + \mathbf{J}(bx)$, is also a case of abnormal flow. One can also imagine systems employing a number of emitting cathodes at different potentials which will exhibit abnormal flow.

The achievement of solutions of normal flow should be much simplified by the theorems given below.

Theorem 2 (Normal Flow)

If the flow is normal, the velocity vector satisfies the relation

$$\mathbf{v} \times \text{curl } \mathbf{v} = 0. \quad \dots\dots(16)$$

Proof. For normal flow $\mathbf{v}^2 = 2\eta V + K$ where K is an absolute constant. Hence $\text{grad } \mathbf{v}^2 = 2\eta \text{ grad } V$. Now, by vector theory,

$$\text{grad } \mathbf{v}^2 = 2(\mathbf{v} \cdot \nabla)\mathbf{v} + 2\mathbf{v} \times \text{curl } \mathbf{v}, \quad \dots\dots(17)$$

where ∇ is the nabla operator $\mathbf{I}(\partial/\partial x) + \mathbf{J}(\partial/\partial y) + \mathbf{K}(\partial/\partial z)$. Hence

$$(\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v} \times \text{curl } \mathbf{v} = \eta \text{ grad } V.$$

But from equation (8)

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{A}, \quad \dots\dots(18)$$

and from equation (13)

$$\eta \text{ grad } V = \mathbf{A}. \quad \dots\dots(19)$$

Hence $\mathbf{v} \times \text{curl } \mathbf{v} = 0$.

Theorem 3 (Converse of Theorem 2)

If $\mathbf{v} \times \text{curl } \mathbf{v} = 0$, then the flow is normal.

Proof. From equations (18) and (19) $(\mathbf{v} \cdot \nabla)\mathbf{v} = \eta \text{ grad } V$. Substituting in equation (17) gives $\text{grad } \mathbf{v}^2 = 2\eta \text{ grad } V$. Hence $\mathbf{v}^2 = 2\eta V + \text{constant}$, which proves the theorem.

Theorem 4 (Normal Flow)

If we choose any vector $\mathbf{v}$ satisfying $\mathbf{v} \times \text{curl } \mathbf{v} = 0$, then the relation $\text{curl } \mathbf{A} = 0$ is automatically satisfied.

Proof. In this case, equation (17) gives $2\mathbf{A} = \text{grad } \mathbf{v}^2$, and hence the result follows.

Theorem 5 (Normal Two-dimensional Flow)

For normal two-dimensional flow $\text{curl } \mathbf{v} = 0$.

Proof. In this case, $\text{curl } \mathbf{v}$ is perpendicular to the plane of $\mathbf{v}$, and hence the result follows from $\mathbf{v} \times \text{curl } \mathbf{v} = 0$.

Application of Theorems 1-5

If one is interested exclusively in normal flow solutions, one may take any velocity vector $\mathbf{v}$, with its components certain functions of the coordinates containing arbitrary parameters, and satisfying the relation

$$\mathbf{v} \times \text{curl } \mathbf{v} = 0. \quad \dots\dots(20)$$

Then by Theorem 4 the first of the two fundamental conditions (9) or (11) is automatically satisfied, and only one more scalar equation, namely (10) or (12), remains to be satisfied by the adjustment of the arbitrary parameters.

In particular, for two-dimensional flow, it follows from Theorem 5 that it is merely necessary to take for $\mathbf{v}$ the gradient of any scalar function containing arbitrary parameters to adjust the latter to satisfy either condition (10) or (12).

Equation (20) shows that a limited set of solutions can be obtained in this way even for three-dimensional flow.

In actual applications condition (10) or (12) usually leads to cumbersome expressions. These can be simplified by the methods of classical potential theory as follows.

Take any solution, or sum of solutions, ϕ of classical potential theory containing arbitrary constant parameters (e.g. coefficients, scale-factors etc.) and equate the velocity vector $\mathbf{v}$ to its gradient, $\mathbf{v} = \text{grad } \phi$. Then it is seen that condition (11) is automatically satisfied. Since now $\text{div } \mathbf{v} = 0$, condition (12) simplifies very considerably to

$$\mathbf{v} \nabla^2 \mathbf{A} = 0. \quad \dots\dots(21)$$

For example, one might take the classical solution (Maxwell) for the potential in the case of a grid between two parallel conducting planes, and determine whether, by adding to it some other simple solution of Laplace's equation, equation (21) may be satisfied. The result would probably give a nearly rigorous solution of current flow in a plane triode, which has never yet been achieved.

Theorem 6 (Scaling Laws)

In an electronic device, if all spacings are multiplied by a factor s and all voltages by a factor w , then the following quantities are multiplied by the factors indicated:

Total current (I_{tot}) by $w^{3/2}$

Transit times (t) by $s/\sqrt{w}$

Velocities ($\mathbf{v}$) by $\sqrt{w}$

Charge density (ρ) by w/s^2

Current density ($\mathbf{i}$) by $w^{3/2}/s^2$

Electric field ($\mathbf{E}$) by w/s

Proof. The proof is best carried out by writing down the fundamental equations of electron flow, as developed in I.

$$\begin{aligned}\operatorname{div}(\rho \mathbf{v}) &= 0, & \mathbf{E} &= -\operatorname{grad} V, \\ (\mathbf{v} \operatorname{grad} v_x) \mathbf{I} + (\mathbf{v} \operatorname{grad} v_y) \mathbf{J} + (\mathbf{v} \operatorname{grad} v_z) \mathbf{K} &= -\eta \mathbf{E}, \\ \operatorname{div} \mathbf{E} &= \rho / \epsilon_0, & \mathbf{i} &= \rho \mathbf{v}, \\ I_{\text{tot}} &= \int_{\Sigma} \rho \mathbf{v} \cdot d\mathbf{S},\end{aligned}$$

where Σ indicates any surface, and noting that they are invariant under the given scale changes.

This theorem is probably quite well known in the form it takes when split up into two similitude relations.

It is of course assumed that factors like secondary emission do not play an appreciable part in the device.

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A Knife-Edge Balance for Weighings of the Highest Accuracy

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ABSTRACT. A detailed description is given of the balance used at the National Physical Laboratory for the most accurate weighings made in the comparison of fundamental standards of mass. Among the several unique features of this balance are the design of the beam, stirrups and pans, the use of recognized methods of geometrical constraint for precise location of these parts, the accuracy of construction, particularly of the knives, and the fineness of the positional adjustments of the components.

In technique of weighing, this balance also presents novel features. It is installed in a vault and, once it is loaded with the masses to be compared, the operations of controlling and reading the balance are entirely effected from outside the closed vault, mechanical means being employed for interchanging the loads undergoing comparison. Moreover, the technique of weighing is so arranged that a series of weighings of standards of mass from each arm of the balance in turn is completed without disturbing the contacts between the knives and their bearing planes. Largely as the result of using this technique, an accuracy of ± 0.001 mg., viz. 1 part in 10^6 , has been attained in the comparison of kilogram masses of appropriate stability.

§ 1. INTRODUCTION

THE balance described in this paper has been designed and constructed at the National Physical Laboratory for use chiefly in the comparison of standards of mass of the highest quality, such as the principal standards of this country. Much progress in precision weighings of this nature was made towards the end of the nineteenth century and descriptions of knife-edge balances incorporating special refinements for weighing standards of mass have been

published by some of the laboratories undertaking this kind of work in other countries. A description of a special form of Rueprecht balance used by the Bureau International des Poids et Mesures at Sèvres, near Paris, was published by Marek (1881) and an account of a Stückrath balance used at the Physikalisch-Technische Reichsanstalt was given by Scheel in 1911. An accuracy of the order 1 part in 10^9 of the standards weighed was claimed for both these balances; for the Stückrath balance it was further stated that this precision was attainable without resort to a large number of weighings.

In subsequent years there appears to have been no substantial improvement in the accuracy attained in weighings of this nature, though it was demonstrated by Heyl (1924) that changes in weight of the order 1 part in 10^9 could be determined by following a special technique recommended by Poynting (1878) in the use of the balance. In the relative weighings made by Heyl, however, there was no necessity to interchange the loads undergoing comparison.

In the balance to be described, an accuracy of ± 0.001 mg. has been attained in the comparison of standards of mass 1 kg. or less, by the method of double weighing.

§ 2. GUIDING PRINCIPLES

The accuracy attainable with a knife-edge balance depends ultimately on the positional stability of its knives. The attainment of accuracy of the order 1 part in 10^9 in the comparison of masses calls for exceptionally fine constancy in this respect. For example, with a beam having arms of length 15 cm., the relative lengths of the arms should be constant to within 2×10^{-8} cm., which is a quantity of the order of atomic dimensions. From consideration of the thermal expansion of the beam, which in the absence of a suitable material of low coefficient may be assumed to be of brass, the relative temperatures of the arms of the beam should be consistent to well within 0.0001° C. The fineness of these fundamental conditions leads to a number of particular refinements such as the following:

(i) *Mechanical.* The beam should be of such rigidity that the relative lengths of its arms do not vary very appreciably when the load is changed.

The form of the knives and their attachment to the beam should be designed to give maximum stability to the relative lengths of the arms of the balance. The highest quality should be attained in the straightness of the knife-edges and the flatness of the bearing planes.

The stirrups should be so designed that the load on a terminal knife is always applied in precisely the same manner.

(ii) *Thermal.* In order to reduce to the minimum any thermal errors that might be incurred in the weighings, the balance should be installed in a situation where the conditions of temperature are very steady, and there must be no interference with the temperature of the balance while the weighings are in progress. This necessitates designing the balance so that the loads under comparison can be interchanged without opening the balance, and also arranging for all the controls of the balance to be operated from a distance.

(iii) *Technique.* It is also necessary to adopt a special technique for carrying out the weighings. In ordinary circumstances a knife-edge balance would be fully arrested between successive determinations of the equilibrium point, and also whenever the loads on the pans are being changed. During arrest the knives and their corresponding bearing planes would be separated. Poynting has shown that the mechanical errors associated with the use of a balance can be considerably

diminished by taking the precaution of arresting the balance without separating the knives from the corresponding bearing planes. In this condition Poynting interchanged the loads undergoing comparison. To enable this to be done he devised a form of clamp which was applied to the underside of one of the balance pans, suitably shaped. Subsequently Scheel, in adopting Poynting's principle, applied it to the suspension stirrups and not to the pans. More recently Heyl employed Poynting's form of clamp for certain weighings in gravitational work in which the highest precision was required.

To demonstrate the necessity for this technique some experiments were made with a simple form of balance beam having a fulcrum knife of high quality, but no terminal knives. From observations of the consistency of the equilibrium point and a knowledge of the mass of the beam and its sensitivity, it was deduced that the position of the fulcrum axis relative to the beam did not vary laterally by more than 8×10^{-9} cm. between successive determinations of the equilibrium point, provided that the beam was brought to rest without breaking the contact at the knife-edge. But whenever the contact was deliberately broken in the experiments, the lateral shift of the fulcrum axis relative to the beam varied by amounts ranging from 25×10^{-9} to $1,000 \times 10^{-9}$ cm., depending on the form of support of the beam and other circumstances.

In the new balance, therefore, provision was made for the use of Poynting's principle. It was not, however, applied in the manner adopted by Poynting and others who have used it. Instead, arrangements were made for the *beam* to be *steadied* directly while the stirrups were hanging undisturbed from the terminal knives, and for the pans alone to be fully arrested for the purpose of interchanging the loads under comparison.

After taking into consideration the various requirements, it was decided to design the new balance on a wider basis than usual and to provide for weighings in air and *in vacuo*, and under conditions free from any appreciable magnetic influence. In extending the scope of the balance in this manner there was no intention of carrying out *in vacuo* any comparisons of existing standards which should properly be made under ordinary conditions of atmospheric pressure.

§ 3. DESCRIPTION OF BALANCE

Figure 1 (see Plate) shows a general view of the balance with its cover raised, and serves to illustrate some of the novel features which have been introduced.

(a) *The beam* (Figure 2).

The beam is made from a gunmetal casting and is of a lattice-shaped design strongly braced to give high rigidity, following the recognized principles of triangulation. In addition, the outer portions of the beam are of T-section. The beam has three horizontal seatings which serve for the attachment of the knives, the arms of the balance being 15 cm. long. These seatings are robustly made and have been ground and lapped flat and parallel. A similar horizontal seating is provided at the top of the beam for the attachment of a lens-mirror for reading the balance by the method of optical projection.

Provision is made for the use of two duralumin balls on the beam as riders. For this purpose a V-shaped longitudinal groove was machined in each of the horizontal arms of the beam; these grooves lie in the plane containing the knife-edges and serve as a run-way for the balls which can be propelled, one in each

groove, by means of suitable control rods manipulated from outside the balance. The grooves are intersected by a series of V-shaped transverse notches, machined with great care at a uniform spacing of one-fiftieth of the length of an arm of the balance so that their positional errors were within ± 0.03 mm. Each notch provides a precise location for a rider ball as it is propelled along the arm of the beam, and as the balls are accurately spherical this arrangement has the merit that the centres of gravity of the balls are accurately located with respect to the notches. The balls are of mass 5.0 and 5.5 mg. respectively. It will therefore be seen that the displacement of the balls through one notch corresponds in effect to changes of weight of 0.10 and 0.11 mg. respectively if the balls are used separately; in combination they can be used to produce precisely any change in the poise of the balance from 0.01 mg. to a few milligrams, the lower limit being finer than is normally required.

The controls for the rider balls can also be used to apply suitably shaped small weights to the uppermost stirrups and so enable the sensitivity to be determined independently of the use of the rider balls.

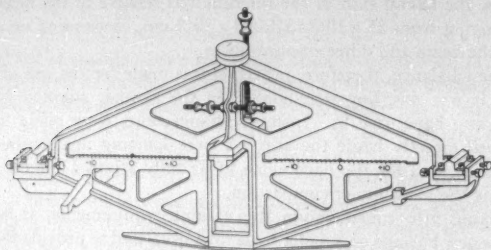


Figure 2. Beam of balance.

When the beam is fully arrested it is supported on three agate points suitably rounded, two on the left and one on the right. For this purpose it is provided with lugs which are machined on the hole, vee and plane principle to give it definite location with respect to the supports. The latter can be lowered or raised by the arresting mechanism; when lowered, the beam rests on a central bearing plane of agate, optically polished. A flexible metal strip (Figure 2), 1.6 mm. thick, is attached horizontally to the lowest part of the beam and is employed for the purpose of holding the beam stationary without fully arresting it. When the balance is swinging, a pair of agate points mounted on an independent bridge below the beam (Figure 1) can be raised so as just to meet the underface of the strip as the beam passes through the horizontal position, and so bring the balance temporarily to rest without disturbing the contacts between the knives and planes. A separate control is provided for this purpose, and the operation can readily be effected without disturbing the bearings. When the beam is in the *steady* condition there is negligible pressure between the agate points and the flexible strip. Provision is made for stabilizing and poising the beam in the usual manner by means of nuts mounted on vertical and horizontal screws attached to the beam. The nuts are locked in position when the desired sensitivity and approximate state of poise have been attained. Fine adjustment of poise is effected by moving the rider balls.

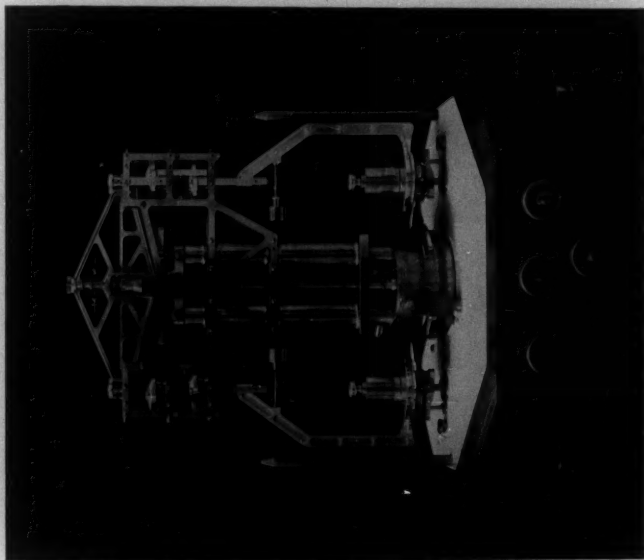


Figure 1. Precision balance, with cover removed.

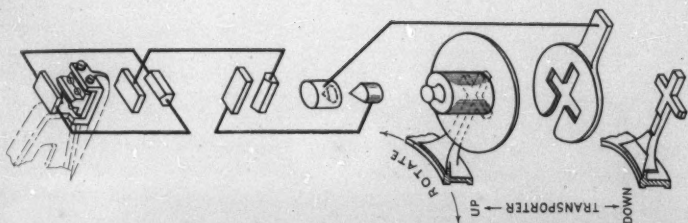


Figure 3. Arrangement of stirrups and pan-hooks, and interchange of loaded pan-plates.

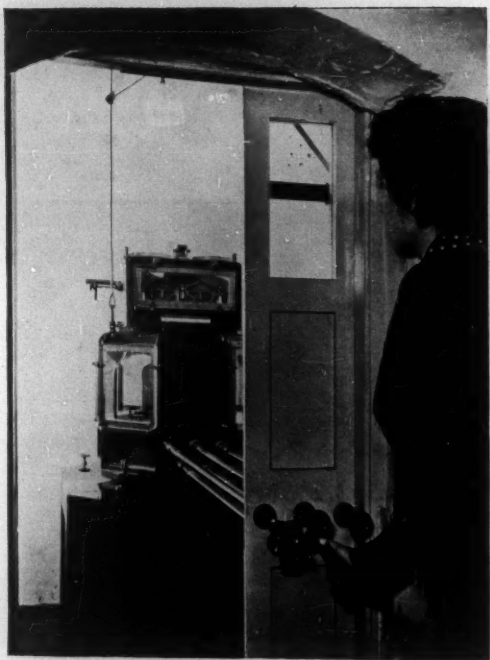


Figure 4. Balance, as installed.

(b) *The knives.*

Figure 2 illustrates the form of the three knives mounted on the beam. The undersurface of each knife is lapped optically flat and is relieved slightly at the middle for mounting on its seating on the beam. Each knife is flange-shaped so as to facilitate attachment to the beam by means of screws, four in number, without causing appreciable distortion of the knife-edge. The facets which serve to define the knife-edge are inclined at an angle of 120° and are ground and lapped optically flat. For this process the knife was mounted in a jig constructed so that the resulting knife-edge is accurately parallel to the base of the knife. Each knife is made of gunmetal similar to the material of the beam. The facets were plated with chromium to a thickness of 0.2 mm. initially, finishing after grinding and lapping at a thickness of 0.1 mm. approximately. Preliminary experiments with knives of this character showed that their hardness was satisfactory and that there was no insuperable difficulty in construction or functioning. The actual knife-edges used were accurately straight and free from abnormality. The fulcrum knife on the beam, which is 44 mm. long, was found to withstand the maximum loading of the balance satisfactorily for a continuous period of several hours without producing marked changes in rest point. Moreover, when the temperature changes, the beam and knives, if free from strain, expand or contract together at a similar rate, and this is an advantage in securing a low temperature coefficient of the rest point of balance.

It is of course well known that steel is in many respects very suitable for the construction of knives of the highest quality, but for the purpose of the new balance it was decided not to use this material owing to its magnetic qualities and its liability to rust.

A parallel-faced packing strip of gunmetal 3 mm. thick was inserted between each terminal knife and the beam in order to provide means for setting the three knife-edges in the same plane, so rendering the balance equally sensitive, to a first order of approximation, for all states of loading. Provision is made for adjusting the terminal knives into parallelism with the fulcrum knife by means of horizontal screws of suitably fine pitch.

(c) *Stirrups and pans.*

The load on the balance is transmitted to the beam through a system of three stirrups illustrated diagrammatically in Figure 3 (see Plate). The first, i.e. uppermost, stirrup is suspended by its bearing plane from a terminal knife on the beam and carries at its lower end an auxiliary knife parallel to the knife on the beam. The second stirrup is suspended from the first and also has a bearing plane and auxiliary knife, but the latter is at right angles to the other knives. The third stirrup hangs by a plane from the second and carries at its lower end a suitably rounded agate point from which the pan-hook is suspended by means of a conical agate cup.

In applying the principle of crossed knives to the design of the first two stirrups so as to locate the load precisely on each terminal knife of the beam, the following considerations were taken into account. The purpose of the first knife of the compensating pair, i.e. that parallel to the knives on the beam, is to avoid variation in the effective length of the arm due to angular changes in the 'hang' of the uppermost stirrup from a terminal knife whose edge is not ideally

sharp. To obtain the best results this compensating knife should not be too close to the terminal knife. The purpose of the transverse knife of the pair is to maintain unaltered the distribution of load along the length of a terminal knife on the beam. The greater the deviation of the latter knife from parallelism with the fulcrum knife, the greater the need for the transverse knife. In theory, according to Thiesen (1886), its ideal position would be at the level of the knives on the beam, but this would give rise to difficulties in design. Above all, it is essential that the construction of the knives, stirrups and planes, and the positional adjustment of the knives, stirrups and supports should be as nearly perfect as possible. In the design eventually adopted, chief reliance was placed on securing these essentials, and particular attention was paid to the efficiency of the uppermost stirrup.

To minimize wear on the standards of mass which will be weighed on the balance, two loose pan-plates are provided which fit into recesses in the bases of the pan-hooks, and the standards when placed on these are not disturbed during the whole of a sequence of weighings. Beneath the plates each pan-hook has an aperture cut in it in the form of a cross, open towards the centre of the balance and designed to permit of the interchange of the pans without opening the balance cover (Figure 3). To effect this, a ring surrounding the raised centre of the base of the balance and bearing two arms with suitably shaped ends which can pass through the apertures in the pan-hooks is first raised, lifting the plates off their supports, then rotated through 180° , and again lowered. Excessive rotation of the hooks about a vertical axis is prevented by suitable stops. The interchange of the pans is effected while the balance is in the *steady* condition with all its knives in contact with their respective bearing planes. The pan-hooks are *arrested* and so held clear of the stirrups for the purpose of the interchange. The stirrups and pan-hooks are made principally of aluminium alloy; the bearing planes and auxiliary knives are of agate. The pan-plates are of polished nickel-chromium alloy (80% nickel, 20% chromium). Corresponding members of the left and right series are adjusted to be closely equal in mass.

(d) *Supports for beam, stirrups and pan-hooks.*

When the balance is fully arrested, the beam and the respective stirrups are supported, each from three points, on a brass carriage which is mounted on two fixed vertical posts, situated at the centre of the balance (Figure 1). The carriage can be lowered by an appropriate amount when it is desired to use the balance, and this operation sets the fulcrum knife on its bearing plane and also in turn suspends all the stirrups from their respective knife-edge bearings. The supports for the beam and stirrups, and the bearings of the carriage against the posts, are all arranged on the principle of geometric constraint so as to locate the component parts of the balance in the same relative positions. The pan-hooks when arrested stand on 3-point supports independent of the carriage and rest with their suspensions just clear of their respective stirrups. Whenever it is desired to suspend the pan-hooks from the stirrups, the 3-point supports are lowered and brought out of contact with their undersurfaces. In order that this arrangement should function satisfactorily for different positions of the loads on the pans, the undersurface of each hook is shaped to form part of a sphere whose centre is at the point of suspension of the hook.

(e) Controls for operating the balance.

Suitable controls, five in number, are provided for the separate stages of releasing or arresting the balance, steadying it and interchanging the pan-plates. These controls are connected to a mechanism situated in the base of the balance, through which the respective operations are effected. Eccentric wheels, mounted on the axes of four of the controls, govern the requisite vertical movements entailed in these operations, viz. the lowering or raising of the pan-hooks, the beam and stirrups, the pan-plates, and the stops which serve to steady the beam. The fifth control provides for the interchange of the pan-plates. A simple device for interlocking the controls is installed to prevent them from being used in the wrong sequence. Extension rods, not shown in Figure 1, are fitted to the controls to enable them to be operated from a distance by an observer situated outside the balance chamber.

(f) Balance cover.

The balance cover is made of a gunmetal casting in the form of a shell 1.3 cm. thick enveloping the balance, and is so constructed that it can form an airtight joint with its base if weighings at pressures other than atmospheric are required. The cover is provided with a number of windows of plate glass which can also be made airtight when required. For this purpose all the seatings for airtight joints have been ground and lapped flat. All the controls for the balance (including those for setting the riders) pass through oil-sealed glands. A hydraulic hoist is installed for raising the balance cover whenever it is desired to open the balance, e.g. for loading.

(g) Non-magnetic character of balance.

Special care was taken to avoid using magnetic material in the construction of the balance, and stringent tests of magnetic permeability were carried out before accepting the materials actually used. These tests were also extended to a sample knife made from gunmetal plated with chromium. For the thickness of plating employed it was verified that the knife was effectively non-magnetic to well within the limits required.

§ 4. INSTALLATION

The balance is mounted on an anti-vibrational pedestal in a vault situated near the centre of a large basement floor (Figure 4, see Plate). The pedestal for the balance is constructed of proved non-magnetic materials and precautions have been taken to exclude magnetic objects from the vault. The natural distribution of temperature within the vault is particularly steady and uniform, the walls being very thick. The uniformity of the temperature of the beam is further increased by the protection afforded by the robust balance cover. The excellence of these thermal conditions has since been confirmed by the high accuracy attained with the balance. In the circumstances there was no need for automatic thermostatic control, but had this been necessary it would not have been easy to provide the high degree of thermal uniformity necessary for the balance to develop its full capabilities.

An optical system is employed for reading the balance on a scale at a distance of six metres outside the vault. An auxiliary scale, on which an index of the balance reading is also projected, is set up in front of the operator when standing just

outside the vault, to assist him in timing the manipulation of the controls. Thus full provision is made for weighing standards of mass from both arms of the balance in turn without entering the vault, once the balance has been loaded and poised. During the weighings the vault is kept dark, apart from the beam of light used for reading the balance.

§5. ADJUSTMENT OF BALANCE

It will, of course, be realized that many adjustments had to be made in order to put the balance in its final condition for weighing. The procedure adopted for the more important of these was as follows:

(a) *Beam.*

To begin with, the fulcrum knife was mounted in a central position on its seating on the beam; then the terminal knives were set provisionally at a distance of 15 cm. from the fulcrum knife, and approximately parallel to it. As explained in the description of the balance, the design and construction of the knives and their seatings on the beam were arranged so that the three knife-edges should be accurately parallel to one another when viewed in elevation from the left or right of the balance. The attainment of high precision in this respect was subsequently confirmed by a mechanical test on the finished beam, in which the heights of the knife-edges above a standard reference plane were accurately gauged. The packing plates under the terminal knives had provisionally been made of a suitable thickness (slightly greater than that finally required) to provide a means of adjustment of the knife-edges into the same plane. At this stage a preliminary determination of the sensitivity of the balance under different conditions of loading was carried out, and from the results so obtained the amount by which the fulcrum knife-edge lay below the plane through the other two knife-edges was computed. By reducing the thickness of one or both packing plates a final condition was reached in which the sensitivity of the balance was of the same order of magnitude for all states of loading.

In order to set the terminal knives parallel (in plan view) with the fulcrum an improvised testing stirrup, provided with a very short bearing plane, was weighed from different positions along the edge of a terminal knife in turn, following a normal procedure. This stirrup was poised against a stirrup of similar mass bearing on the whole length of the other terminal knife, and it will readily be seen that the results of such weighings afford a measure of the straightness of the first knife-edge and also of its parallelism with the fulcrum knife-edge. The position of each terminal knife was adjusted by stages to give the closest parallelism attainable. Its lateral adjustment was made by means of two pairs of fine horizontal screws bearing on the sides of the flanges of the knife (Figure 2). During the adjustment the four vertical screws which serve to fix the knife to the beam were released slightly. At the conclusion, when these were fixed, the horizontal screws were slightly withdrawn from the flanges so that the knife was not left in a condition of strain.

This adjustment of parallelism was made concurrently with the adjustment of the lengths of the arms, using the same horizontal screws. The lengths were adjusted to be equal to fifty times the mean pitch of the notches comprising the rider scale. For the purpose of determining the linear relationship between the arms and the notches a suitably shaped weight, specially made for the purpose, was placed in a notch near one end of the series and poised against a mass suspended from the end of the other arm of the balance. On repeating this procedure, but

setting the weight in a notch near the other end of the series, the requisite information was obtained. In making the final stages of adjustment of the terminal knives, more importance was attached to the attainment of a high degree of parallelism of the knives than to the exact equalization of the lengths of the arms, but good precision in the latter was eventually obtained.

The centre of gravity of the beam was finally adjusted, by means of the nut provided for the purpose, so that the balance was sufficiently sensitive to enable the equilibrium point to be determined with a precision well within ± 0.001 mg. In this condition the period of a complete swing was found to be about 70 sec. with no load on the pans, and 120 sec. with 1 kg. on each pan.

(b) Stirrups.

During a series of weighings it is essential that the 'hang' of the first (top) stirrups on the beam should not be affected by the interchange of the loaded pan-plates or other causes. Each of the combinations of three stirrups on the left and right of the balance may be regarded as providing two universal joints through which the load is transmitted to the beam, viz. (i) the pair of crossed knives formed by the first and second stirrups in combination; (ii) the conical agate point on the third stirrup from which the conical cup of the pan-hook is suspended.

In fixing the positions of the stirrups it was highly important to set the two universal joints of the same series in a vertical line passing through the bearing edge of the corresponding terminal knife on the beam, and to retain this setting whether the balance be arrested or released. In the arrested condition of the balance each stirrup is suspended from three points on the arresting carriage. These points take the form of conical-headed screws which can be raised or lowered to the required levels and can also be traversed according to the nature of the positional adjustment required. In view of the unsymmetrical shape of the second and third stirrups, they were constructed so that the centre of gravity of each stirrup was adjustable to the desired position, and this was an essential preliminary to the positional adjustments of the supports for the stirrups. The procedure of adjustment was as follows. With only the first stirrup mounted on each arm of the balance, the supporting points were adjusted in position so that the knife-edge on the stirrup lay in the vertical plane through the corresponding knife-edge on the beam, irrespective of whether the stirrup was arrested or released. The centre of gravity of the stirrup therefore lay in this vertical plane. The second (i.e. intermediate) stirrups were next mounted on the balance and their supports were adjusted in position until their knife-edges were accurately horizontal in both the arrested and released conditions of the balance, the centre of gravity of each stirrup therefore lying in the vertical plane through a terminal knife on the beam. The third (i.e. lowest) stirrups were next mounted on the balance and their positions set so that the agate points at their bases lay vertically below the knife-edges on the first and second stirrups respectively, in both the arrested and released conditions of the balance. These adjustments were completed after much labour in which measurements of verticality and horizontality by micrometer telescopes were referred to suitable reference standards, e.g. a fine plumb line and a liquid level. Before finally locking the supports for the stirrups, they were set so as to allow a uniform parallel clearance of 0.03 mm. between each knife-edge and its bearing plane when the balance is arrested; this is about half the clearance allowed at the fulcrum knife.

At the close of the adjustments there was scarcely any perceptible lateral movement of the stirrups upon releasing or arresting the balance, or after transposition of the loads. In a final test, carried out under the working conditions of use of the balance, the maximum amount of tilting movement of the uppermost stirrups about the terminal knives on the beam was found to be less than 10^{-5} radian.

(c) *Pan-hooks.*

By means of the balance weight provided, each pan-hook was adjusted so that its centre of gravity lay in the vertical line through the centre of the pan-plate. If the standards of mass undergoing comparison on the balance are carefully centred on the pan-plates, their interchange can be effected with the minimum of disturbance to the pan-hooks and their suspensions. With this object in view the upper faces of the plates are marked with a series of concentric rings, lightly burnished in the polished metal, so as to facilitate centring the standards on the plates. The accuracy of centring can readily be checked by raising the plates temporarily and observing whether the pan-hooks take up a new plumb position when hanging freely. When the balance is fully arrested, a clearance of 0.03 mm. approximately is allowed at the cone-and-cup bearing at the top of each pan-hook. The amount of clearance is controlled by the adjustable positions of the three-point supports on which the pan-hooks rest.

(d) *Steadying of beam.*

In order to avoid any disturbance of the positions of the stirrups on the beam, the agate stops which serve to steady the balance were adjusted in height so as to permit a minute but appreciable amount of swing of the beam when in the *steady* condition. It was found convenient to allow the index of the balance to wander over a range of nearly a centimetre on the reading scale. In these conditions it is easy for the operator to manipulate the control so that the beam is not steadied until the index lies within the range of freedom provided for the purpose. With the control applied in this manner, the index soon comes to rest, and the beam makes contact with a stationary stop instead of being touched by a moving stop.

(e) *Interchange of pans.*

To minimize the possibility of disturbance to the stirrups resulting from the interchange and re-application of the loads undergoing comparison, it is important that the centres of the ends of the transporter arms which engage with the undersides of the respective pan-plates should lie in the lines of application of the loads to the beam. In the construction of the balance, the effective lengths of these arms were made equal to that of each arm of the beam, and the fulcrum knife was set as closely as possible vertically above the centre of rotation of the transporter. A residual error in this setting was corrected by slight lateral displacement of the carriage on which the beam and stirrups are supported when the balance is arrested and by very slight tilting of the base of the balance.

In the final condition of the balance there was scarcely any appreciable swinging of the stirrups and pan-hooks about their supports during a set of weighings.

§ 6. PROCEDURE OF WEIGHING

Before discussing the performance of the balance, the procedure adopted in weighing will be described. This relates to weighings made under ordinary atmospheric pressures; weighings under reduced pressure would only be made

in special circumstances. Each determination of a difference between two loads extended over two days. In the evening before the first day, the loads about to be compared were placed centrally on the interchangeable pans of the balance and the two combinations of pan and load were equalized to within 0.05 mg. This was done by placing on the pans the appropriate combination of weights of known mass taken from a set having the nominal values 1.00, 1.05, 1.10, 1.20, 1.30 . . . mg. The balance was finally poised so that its equilibrium point corresponded approximately to the centre of the scale.

In the morning of the following day, without entering the inner chamber of the balance vault, the observer brought the beam into a stable condition by carrying out a short series of preliminary weighings. The first comparison of the day was then made, as follows. The balance was set swinging, and the first complete oscillation allowed to pass unrecorded. The equilibrium point was then determined from five readings corresponding to the ends of consecutive swings. The beam was next *steadied*, the pan-hooks alone arrested, and the pan-plates with their loads transposed. After releasing the beam once more the new equilibrium point was determined, as before. In general, nine or more equilibrium points were determined following this procedure. The loads were transposed between successive determinations, the knives and bearing planes remaining undisturbed throughout. On completing the observations, the balance was fully arrested.

Two sets of equilibrium points were thus obtained, corresponding to the weighing of each load from each arm of the balance. While the beam was still swinging after the last reading, the current sensitivity over an appropriate range of scale was found by placing very small weights on the uppermost stirrups. The weighings, excluding the sensitivity test, were repeated the same day. At the end of the day the balance was opened and the loads interchanged on their pans in readiness for the succeeding day, when two similar series of comparisons and a sensitivity determination were made.

The requirements in regard to the steadiness of thermal conditions were extremely exacting. The effect of such minute thermal changes as did occur was exhibited by a drift of the two sets of equilibrium points in a parallel fashion. Owing to the nature of the drift, allowance for it was readily made, and apparently the accuracy of weighing was not limited by thermal changes.

Occasionally a discontinuity occurred in a series of equilibrium points. This was thought to have been due to an infinitesimal positional change in one or more of the stirrups. As subsequent equilibrium points were usually all affected similarly, the discontinuity could easily be recognized and dealt with appropriately.

Each set of weighings was corrected for the difference in atmospheric buoyancy of the loads under comparison. The results of two days' weighings were combined so as to eliminate the difference in weight of the pan-plates, assuming the loads and the pans to remain constant in mass during the weighings.

§ 7. PERFORMANCE

In order to assess the performance of the balance, two kilograms of the highest available stability, made of platinum and platinum-iridium alloy respectively, were weighed repeatedly during four periods of two days each, following the procedure outlined above. The work was duplicated by two independent observers, each of whom completed two sets of observations each day. The differences between the loads compared, as determined by each set, were in almost all cases within ± 0.001 mg. of the mean for the day. Out of 32 such

differences obtained during eight days' weighings, 30 were either zero or 0.001 mg., the remaining two being 0.002 mg. The difference between the two kilograms was determined, as explained, by combining corresponding sets in each period of two days. Consistent values were obtained for this difference during the first six days, the twelve individual values being, with one exception, within ± 0.001 mg. of the mean. The four values for the last two days, however, although consistent among themselves to well within ± 0.001 mg. of their mean, differed by 0.005 mg. from the earlier values. This change in value was attributed to the effects of changes in mass of one or both kilograms and pans, but these effects could not be separated adequately, although comparisons of the unloaded pans were also made at successive stages of the work. At this point further study of the performance of the balance was put aside owing to the exigencies of war.

In 1947 the work was resumed and intercomparisons of four standard kilograms of platinum or platinum-iridium were made, two at a time, in all six possible combinations. The difference between two kilograms was determined from weighings made on two successive days, but the number of sets of weighings on each day was reduced to two, viz. one by each of two observers. As in 1940, there was evidence of a small change in mass of one of the kilograms and one or both pans. The six values of the differences between the kilograms of each combination were closely interconsistent with the exception of the sixth combination, for which it was found and confirmed by subsequent weighings that a small change in mass had occurred. With the exclusion of this combination, the remaining five values showed departures from perfect interconsistency which were in the vicinity of 0.001 mg., the largest being less than 0.002 mg. Later in 1947, a series of intercomparisons of six platinum pounds was completed, and similar results obtained. Small changes in mass of one of the pounds and one or both pans were found to have occurred at one stage, at least, of the weighings, the change in the pound predominating.

The accuracy of ± 0.001 mg. attained by the balance described in this paper corresponds to one part in a thousand millions in the comparison of kilograms. This accuracy is unprecedented in the comparisons of standards of mass.

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Onset of Turbulent Flow in certain Arrays of Particles

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ABSTRACT. Reynolds' number has been found for the onset of turbulence in water flowing through a perfectly packed hexagonal array of uniform spheres. Experiments have also been carried out on an imperfectly packed array of the same spheres and on an array of irregular particles (marble chips). A satisfactory differential manometer, which enables differences of head of the order of a hundredth of a millimetre of water to be readily determined, is described.

§1. INTRODUCTION

DURING an investigation of the mechanism of dilatancy (Andrade and Fox 1949), information was required concerning the onset of turbulence in water flowing through an ordered array of uniform spheres in contact with one another. A search of the literature proved fruitless. Although figures were available for assemblies of irregular particles (sand and such like) and even disordered arrays of uniform spheres, none were found for the perfect array. Accordingly, experiments were performed to determine the onset of turbulent flow in a hexagonal array of uniform spheres. A few experiments were also carried out on assemblies of imperfectly packed spheres and on irregular particles, i.e. marble chips.

§2. EXPERIMENTAL ARRANGEMENT

The experimental arrangement is illustrated in Figure 1. Water from the constant head tank T, flows through the system at a uniform rate, leaving at E. The array is contained in the brass tube H, which, for reasons given below, is of hexagonal cross section. The pressure difference across the array is measured by the manometer M, in which sensitiveness is attained by using two immiscible liquids of nearly equal densities.

The apparatus was designed to permit easy filling with water and removal of air bubbles trapped in the interstices of the array. It was found to be essential to remove all these bubbles in order to secure consistent results.

The volume of water flowing through the array in a given time was determined by means of a measuring cylinder and stop-watch, different rates of flow being obtained by adjustment of a screw-clip at the exit of the system. The temperature of the water was taken immediately after each collection. The heights of the menisci in the manometer limbs were determined by a cathetometer, which could be read to 0.005 cm.

§3. PACKING OF ARRAYS

The simplest close-packed medium in three dimensions is an array of uniform spheres arranged in hexagonal formation. Such an array can be built up by hand in a tube of hexagonal cross section.

The spheres were $1/8$ in. diameter, being balls used in bearings. They were made of phosphor bronze which does not rust. The bottom layer of balls of the array was supported on 7 wires, $1/32$ in. diameter, suitably disposed in a

cross section of the hexagonal tube. Thus the wires all ran parallel to two opposite sides of the hexagon and were spaced with their axes at distances of

$$(1 + n\sqrt{3})/16 \text{ in.}, \quad n = 0, 1, \dots, 6,$$

respectively from one of these sides. The remaining balls were piled on the bottom layer and thus the array was supported with the minimum of constraint. The balls in any layer were exactly over the balls in the next layer but one. There were 37 balls in the bottom layer, 27 in the next layer and so on alternately, with 37 in the top layer. The hexagonal tube just fitted into brass tubes of circular cross section, one at each end, the abrupt changes of cross section from hexagonal to circular within the tubes being smoothed over with a hard-setting cement. Each circular tube carried a side tube, bent at right angles as shown in Figure 1, on which the manometer limbs were mounted.

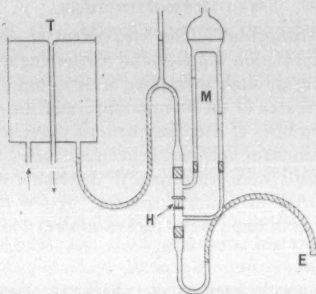


Figure 1. General experimental arrangement. The shaded portions represent rubber tubing.

An imperfectly close-packed array of the balls was formed by 'pouring' them into the hexagonal tube and gently shaking them down. Sieved marble chips were used as irregular particles, those particles being selected which passed a mesh of 0.125 in. opening and were retained by a mesh of 0.128 in. They were also closely packed by 'pouring' and gently shaking down.

§4. THE MANOMETER

The second liquid used in the manometer was phenetole. It was carefully introduced into the flask by pouring it down a glass rod until the water level in the limbs was depressed to a position corresponding to that in the left-hand limb of Figure 1. A flow was then established and the manometer behaved as though it contained a single liquid of density equal to the difference of the densities of water (ρ_W) and phenetole (ρ_P). Accordingly, it gave a magnification of

$$\rho_W / (\rho_W - \rho_P),$$

as compared with a simple water manometer. Water and phenetole give a magnification of about 30, so that pressure differences down to 1 dyne cm⁻², corresponding to a 'head' of 0.3 mm., can readily be determined.

One liquid which has been used with water, namely castor oil, was tried first, but was rejected owing to its sluggish action. A search was made in the tables for an alternative liquid and phenetole was chosen. Its density at 20°C. is 0.967 gm.cm⁻³, as compared with 0.960 gm.cm⁻³ for castor oil. Thus it gives

a magnification 20% greater than that given by the oil. The real advantage, however, is in mobility, the respective viscosities of phenetole and the oil at 20° c. being 0.0126 and 9.9 poise. Phenetole also gives a very good meniscus against water, being slightly superior to castor oil in this respect. It appears to take up a little water in time, becoming slightly cloudy in the process. This presents no difficulty as the water is readily removed by calcium chloride and distillation.

The manometer limbs were precision bore 'Hysil' glass tubes made by Messrs. Chance under the trade name 'Veridia'. The internal diameters were 1 cm. with a tolerance of ± 0.001 cm. Hence any difference of levels due to surface tension effects was negligible.

It was essential that the manometer limbs were scrupulously clean and that water-free phenetole was used for each determination of a critical velocity. During an experimental run the rate of flow was steadily increased from experiment to experiment, since, on decreasing the rate, the menisci tended to invert, becoming distorted and thus rendering impossible an accurate determination of the 'head'. The last difficulty probably accounts for the neglect of the two-liquid manometer in practice.

§ 5. RESULTS

To deduce a critical velocity for the onset of turbulence, points were plotted with log (head) as ordinate and log (rate of flow) as abscissa. Two typical curves are shown in Figure 2. The upper refers to 15 layers of the hexagonal array and the lower to 7 layers. These two quantities were used to make sure that the

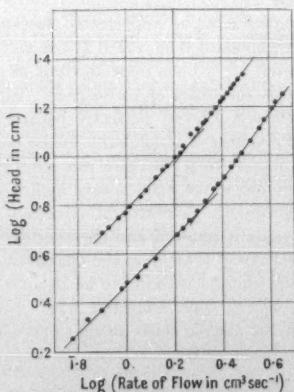


Figure 2. Flow of water through a hexagonal array of $\frac{1}{8}$ in. diameter phosphor bronze spheres. The lower curve refers to an array of 7 layers of spheres, the upper to 15 layers.

critical velocity was determined for the medium in bulk. It will be noted that the laminar region is represented by a straight line of slope unity. This is in accordance with Darcy's law (see, for example, Muskat 1937), since the head is proportional to the pressure difference across the array and is, therefore, proportional to the rate of flow, provided the flow is laminar. Moreover, the two

lines of slope unity are displaced by 0.3125, so that $\log$ (head for 15 layers/head for 7 layers) = 0.3125 for any rate of flow in the laminar region. The ratio of the two heads is, therefore, 2.05:1, whereas the ratio of the lengths of the arrays is 15:7 or 2.14:1. Thus this result sensibly illustrates Darcy's law.

The turbulent region in Figure 2 is also represented by a straight line, but of slope greater than unity. This linearity exists because the range of rates of flow considered is small. As we advance further into the turbulent region, the resistance offered by the array increases at a higher power of the rate of flow than that represented by the slopes of the lines in Figure 2. For example, the results of an experiment on 15 layers of the perfect array, using a water manometer, can be adequately represented by three lines of slopes 1.23, 1.49 and 1.62 corresponding to ranges of rates of flow from about 1.5 to 5.0 cm³ sec⁻¹, 7.6 to 16.8 cm³ sec⁻¹ and 16.8 to 30.0 cm³ sec⁻¹ respectively. No attempt was made to explore the turbulent region in any greater detail.

The intersection of the lines in Figure 2 gives a critical rate of flow for the onset of turbulence within the hexagonal array. Dividing this by the area of cross section of the hexagonal tube (3.68 cm²) gives a critical velocity, which is to be interpreted as the mean velocity of the water as it reaches the first layer of balls. The results of two independent experiments on 7 layers of balls were 0.54 and 0.57 cm. sec⁻¹, whereas with 15 layers, 0.52 and 0.60 cm. sec⁻¹ were obtained. The agreement is reasonably good, and a mean value of 0.56 cm. sec⁻¹ is therefore adopted for the hexagonal array. Taking the diameter of the balls as a characteristic linear dimension of the system and the kinematic viscosity of water at atmospheric temperature as 0.0100 cm² sec⁻¹, we obtain a Reynolds' number of approximately 18.

Two independent experiments on imperfectly close-packed arrays of the ball bearings give critical velocities of 0.65 and 0.71 cm. sec⁻¹. The corresponding Reynolds' number is about 22. Thus imperfections of packing appear to make little difference as regards the onset of turbulence. 229 balls were used, this being the number to form a 7-layer, perfectly packed, hexagonal array. No estimate was made of the degree of irregularity of the packing, since the experiments were carried out merely to see if there was a big effect.

In the case of marble chips, the results were inclined to be rather irregular. Three independent experiments gave critical velocities of 0.25, 0.27 and 0.17 cm. sec⁻¹. If the last value is ignored and the average grain size, as estimated by 'go' and 'not go' in sieving, is taken as the characteristic linear dimension of the system, the Reynolds' number is found to be about 8, roughly one third that for imperfectly packed balls of about the same size. No doubt turbulence sets in at a lower velocity for the marble chips owing to the sharp edges of the grains and, for the same reason, the critical velocity is more sensitive to inevitable differences of packing.

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A Phenomenological Theory of Sintering

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ABSTRACT. Sintering occurs when powders are heated to temperatures near their melting points. This paper deals with the rapid increase of density during the sintering of single substances. The increase of density cannot be explained by volume diffusion of vacant lattice sites or surface migration of atoms, but must involve macroscopic flow. The driving force for this flow is surface tension, and an equation connecting the rate of shear strain with the shear stress defines the resistance to deformation.

The density of a compact is calculated as a function of the time for two different laws of deformation, (a) that for a solid with a Newtonian viscosity, and (b) that for a Bingham solid which has a yield point and a rate of shear strain proportional to the difference between the applied shear stress and a yield stress. The effect of gas in the pores is calculated in the case of the viscous law.

The theory assumes that the pores are equal spheres and predicts that densification is uniform throughout a compact, independently of its shape and size, and suggests that gas pressures of a few atmospheres applied to the outside of a compact may appreciably increase the rate of sintering.

Relevant experiments and previous theories are examined critically, and it is shown that while the viscous model may explain the sintering of glasses it cannot explain that of metals. However, the experimental data can be explained by a model showing a yield point: on such a model the interaction of one pore with its neighbours is vital, so that pores in powder compacts close and isolated pores do not.

§1. INTRODUCTION

WHEN powders of metals, ionic crystals, or glasses are heated to temperatures near to their melting points, the powder particles weld together and the density of the compact changes: this process is known as sintering. Sintering has been used from the earliest times for fabricating solid lumps of metals, such as iron or platinum, which could not be melted; today it is still used to fabricate materials of high melting point such as tungsten and alumina. Since about the middle of the last century it has been possible to melt most metals, and although interest in sintering then waned, it has recently increased again because it is sometimes cheaper to press and sinter small awkwardly shaped parts than to cast and machine them. In addition, sintering techniques can be used to produce special products such as porous bearings and, by the inclusion of gas-producing materials in the compacts, expanded glass products. The history of sintering has been reviewed by C. S. Smith (1942).

Sintering is a process which reduces the surface area of the powder particles, and the driving force arises from the excess free energy of the surface of the powder over that of the solid material. The fundamental problem of sintering is to explain by what mechanisms this reduction of energy occurs. In this paper a phenomenological theory of the sintering of single substances is developed; the atomic mechanisms operating remain obscure, but attention is drawn to certain basic phenomena which must be explained, and to the necessity for further experimental work. The literature dealing with the sintering of metals has been reviewed by Rhines (1946) and by Shaler (1949).

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At least two processes occur during sintering: (a) the powder particles weld together and the pores between them become more nearly spherical, and (b) the density of the compact increases. Although these two changes proceed simultaneously and the driving force for both arises from the excess surface free energy of the powder over that of the bulk material, they may occur by different mechanisms. The main theme of this paper is to explain the remarkably rapid increase of density which occurs during sintering. In §2 it is shown that the observed increase of density cannot be due to the volume diffusion of vacant lattice sites or surface migration of atoms, and that probably both volume diffusion and non-uniform flow are responsible for the sphering of the pores. In §3 it is shown that the driving force for sintering is surface tension, and that at high temperatures it is permissible to assume an equation of mechanical state connecting the rate of shear strain of the material with the shear stress at constant temperature.

So that the reader may follow the general trend of the arguments to their final conclusion, the mathematical calculation of the density of a compact as a function of the time is deferred until §5. The calculation is carried out for two different equations of state: (a) that for a solid with a Newtonian viscosity, for which the rate of shear strain is proportional to the shear stress, and (b) that for a Bingham solid which has a yield point and a rate of shear strain proportional to the difference between the applied shear stress and the critical shear stress. The effect of gas in the pores is considered in §6, and some limitations on the possible atomic mechanisms in §7. In §8 some suggestions are made for experimental work.

Shaler and others have made certain experiments which were designed to test Frenkel's theory that sintering is due to viscous flow induced by surface tension. In §4 both the theory and the experiments are examined critically, and it is shown that although the viscous hypothesis may explain the sintering of glasses it cannot explain the sintering of metals. It is indicated in general terms how all the experimental results can be explained if it is supposed that the material shows a yield point. Attention is drawn to the lack of experimental data concerning the mechanical properties of metals at sintering temperatures.

In discussing the work of other authors, the present authors have sometimes drawn different conclusions from the original authors, or stated conclusions which did not appear in the original paper. Therefore many matters have been discussed at greater length than might otherwise have been the case.

§2. SPHERING OF THE PORES

Before proceeding with the main theme it is convenient to show that neither volume diffusion of vacant lattice sites nor surface diffusion of atoms can account for the observed changes in density, and to discuss the experiments relating to the sphering of the pores.

All the lattice points of a crystal in thermal equilibrium at a finite temperature are not occupied by atoms. At the melting point of copper the concentration of vacant lattice sites is about 1,000 times the concentration of atoms in the vapour phase. These vacant lattice sites can move through the lattice and are responsible for self-diffusion. Just as the equilibrium vapour pressure is greater outside a convex surface* than outside a concave surface, so the equilibrium concentration of vacant lattice sites is greater inside a concave surface than inside a convex surface.

* A surface is described as concave or convex according to its appearance to an observer situated in the vapour phase.

Now volume diffusion alone can only change the density of a compact provided that vacant lattice sites diffuse from the interior to the outer boundary. This is a negligibly slow process: for, in order that pores in the centre of the compact should disappear, it is necessary that diffusion should occur over large distances. A simple calculation shows that if the excess concentration of vacant lattice sites in copper at $1,000^{\circ}\text{C}$. is produced by pores of 1μ radius, and diffusion has to take place over a path of 1 mm., then the decrease of linear dimensions due to volume diffusion will be only $3 \times 10^{-6}\text{ cm/day}$; Kuczynski's experiments (1949) indicate that at $1,000^{\circ}\text{C}$. surface diffusion is even less effective in transporting material. Further, if volume diffusion were responsible for the increase of density, no net diffusion to the surface could take place until there were no pores between the surface and the point under consideration. Sintering would then occur from the surface inwards, and the time of sintering would be a function of the shape and size of the compact. This conclusion is the opposite of that drawn in §5 (i) from the theory developed later in this paper.

On the other hand, although diffusion cannot change the density of a compact appreciably, it can move matter over short distances and so change the shape of the pores. Diffusion moves atoms from the high-energy convex parts of the surface to the lower-energy concave parts so that the pores become more spherical and their surface area less. The following phenomena may be explained in terms of a mechanism of this type. Chalmers, King and Shuttleworth (1948) observed that scratches on a silver surface filled up when it was heated in air near to its melting point: they suggested that the most important mechanism for transfer of atoms was surface diffusion. Kuczynski (1949) placed a number of copper and silver spheres ($\sim 100\mu$ diameter) on a plane surface of the same metal and heated them in an inert atmosphere near to their melting points. The crack between the sphere and the plane gradually filled up, and this was attributed to volume diffusion of vacant lattice sites.

Kuczynski derives a relation between the rate of filling up of the cracks and the self-diffusion coefficient of the metal. He shows that, for copper and silver, the self-diffusion coefficient derived from his experiments has the same activation energy as that derived from experiments with radioactive tracers, and has roughly the same magnitude. Kuczynski also claims to be able to distinguish between the mechanisms of volume and surface diffusion by a difference in the dependence of the size of the neck on the time. This is probably not correct as Kuczynski's theoretical treatment of the problem is over-simplified. A new treatment by Cabrera shows that the dependence of the size of the neck on the time is the same for both mechanisms, and that surface diffusion can only be distinguished from volume diffusion by its lower activation energy.* Thus, because of the numerical agreement between the activation energies, the present authors conclude that, in Kuczynski's experiments, the dominant mechanism for transporting matter over short distances at sintering temperatures is volume diffusion. Chalmers, King and Shuttleworth's observation is presumably due to volume diffusion also.

However, it is shown later in this paper that plastic flow and the interaction of one pore with its neighbours are vital to the rapid increase of density during sintering. These two factors were missing in Kuczynski's experiments, so that volume diffusion will not necessarily be the only mechanism causing sphering of the pores during sintering. In fact, differential flow can also lead to sphering,

* The authors are greatly indebted to Dr. Cabrera for privately communicating these results.

and may possibly be the dominant mechanism for sphering, particularly in the final stages. For, anticipating later sections, the inward pressure due to surface tension will be greatest in the neighbourhood of those parts of a pore which have the greatest concavity, and so the greatest flow would be expected in these parts. When a sphere stands on a plane, the shear stress will be almost zero everywhere except in the neighbourhood of the neck. Thus, although the critical shear stress is probably exceeded in the neck and the material there plastic, the bulk of the sphere (and the plane below) will not be able to flow plastically, and the constraint thus imposed will effectively prevent flow occurring.

§3. THE DRIVING FORCE AND RESISTANCE TO DEFORMATION

(i) *The Driving Force is Surface Tension*

Shaler and Wulff (1948) showed experimentally that forces as small as those of surface tension may cause sintering. The surface tension acting in the surface of a pore of radius 30μ in a copper compact is equivalent to a negative (inward) pressure of about one atmosphere inside the pore. Shaler superposed upon the surface forces a gas pressure of one atmosphere and measured the change of density of a compact. An unpressed copper compact was heated in an inert gas until the pores sealed and an equilibrium relative density equal to 82% of the density of solid copper was obtained; in this state, because of the surface tension, the gas pressure inside the pores was greater than that outside. The gas pressure outside the compact was then removed and, under the driving force of one atmosphere pressure, the density decreased to a new equilibrium relative density of 60% in less than eight hours.

Some authors have suggested other forces. In commercial practice metal powders are pressed at 20–30 tons/in² before sintering, and this will cold-work the powder. It has been suggested that the strain energy of cold-work is the driving force for sintering. This is not very likely since, at high temperatures, recovery and recrystallization occur in a few seconds, and any strain energy will be dissipated before it can be used to aid sintering. It should be noted that Shaler's experiment above was carried out with an unpressed powder.

(ii) *The Existence of an Equation of State*

In a compact formed from spherical particles in contact, an increase of density will occur only if the centres of the particles get closer. This can occur either by a substantially radial flow of material into the pores, or by transfer of material away from the areas of contact. Kuczynski's (1949) photo-micrographs seem to indicate that, after a sphere has been heated in contact with a plane for a number of hours, material is transferred from the plane towards the area of contact; the sphere is left standing on a pedestal. From this, and the fact that diffusion is incapable of explaining large increases in density, it is concluded that flow of the material on a macroscopic scale is required to explain increases in density.

During sintering, the temperatures are high and deformation occurs slowly compared with the rates of recrystallization and recovery. The material will anneal as deformation proceeds, its mechanical state will be the same at all times, and no strain hardening will occur. In these circumstances the rate of strain will depend only upon the stress and not upon the strain or the previous history of

the material. The equation which connects the rate of strain with the stress at constant temperature is an equation of mechanical state which defines the resistance to deformation.

While the energy released by recrystallization and recovery is irrelevant for determining the driving force, the rate at which these, or other annealing processes, occur will be factors which determine the rate of strain corresponding to a given stress, i.e. the equation of mechanical state.

The theory of sintering will be developed on the assumption that deformation is due to surface tension and that the material of the particles has homogeneous mechanical properties defined by an equation of state. It will be assumed that purely hydrostatic stresses and strains are unimportant in producing large deformations by flow; thus, the equation of state will be assumed to connect the rate of shear strain with the shear stress; for simplicity in the calculations, the material will also be assumed to be incompressible. The actual form of the equation of state is a matter for future experiments: glass is known to have a Newtonian viscosity, the rate of shear strain being proportional to the shear stress, while it is deduced from sintering experiments that crystalline metals should show a yield-point phenomenon which must be taken into account.

Both W. D. Jones (1937) and Bangham (1947) have previously recognized that sintering probably involved plastic flow.

§ 4. THE KINETICS OF SINTERING

(i) *Experimental Data*

When metals are sintered large increases of density can occur in very short times. A compact of tungsten powder heated to just below its melting point has all its linear dimensions decreased by 17% in 15 minutes (Smithells 1945). Shaler (1949) finds that the relative density of an unpressed powder of copper increases from 50% to 95% after heating for six hours at 930° c., a decrease in linear dimensions of 24%. The melting point of copper is 1,083° c.

Shaler and his collaborators (Shaler and Wulff 1948, Shaler 1949, Udin, Shaler and Wulff 1949) have attempted to develop a theory of sintering based upon Frenkel's ideas (1945) concerning the viscosity of solids. They have performed three important experiments in which they try to show that copper has a Newtonian viscosity at high temperatures.

In the first set of experiments unpressed compacts of spherical copper particles, all of about 110 μ radius, were heated in a vacuum for various periods up to 45 minutes; the density increased by amounts up to 30%.

In the second set Shaler bored cylindrical pores of radius 170 μ along the axis of a wire of radius 1,650 μ and measured the decrease of pore radius after heating to 1,000° c. for 48 hours. The results were erratic, and of three specimens measured the radius of one pore decreased monotonically by 8%, while the radii of the other two pores increased by 5% during the first 24 hours and then decreased to their original value during the second 24 hours.

In the third set of experiments Udin *et al.* suspended a series of weights by means of fine copper wires (radius 36 μ and 64 μ) which were then heated to temperatures between 950° c. and 1,050° c. They found that if there were no weights on a wire it shrank because of the surface tension acting upwards, but if the weight overbalanced the surface tension the wire stretched. They measured the final strain of wires heated for between one and five days, but their technique did not

enable them to measure the extension of one wire as a function of time. At 1,050° C. surface tension acting alone caused the wire to contract 0.4% in one day.

By finding the weight necessary to balance the surface tension Udin was able to calculate the surface tension of solid copper. He gives a value 1.4×10^3 dyne/cm. This value agrees roughly with the theoretical value of the surface free energy per unit area derived by Huang and Wyllie (1949), viz. 1,820 erg/cm².

(ii) Previous Theories of Sintering

Frenkel (1945) supposed that the increase of density during sintering occurs because both amorphous and crystalline materials behave at high temperatures as though they were liquids with Newtonian viscosities; the driving force was supposed to be due to surface tension. He suggested that this viscosity arose because of the existence of vacant lattice sites in thermodynamic equilibrium, and predicted a value of the coefficient of viscosity $\eta = kT/D\delta^3$, where T is the absolute temperature, D is the coefficient of self-diffusion and δ^3 is the volume per atom. Using the data of Maier and Nelson (1942) for D , Frenkel's formula predicts that the viscosity of copper at 1,000° C. is 3.1×10^3 poise*. However, Nabarro (1948) showed that Frenkel's calculation of the coefficient of viscosity was not correct for crystalline materials, and that a crystal which contained no imperfections other than vacant lattice sites would not be viscous. On the other hand, if the crystal had a mosaic structure it would show a Newtonian viscosity, and the coefficient of viscosity would be greater than that given by Frenkel's formula by at least a factor $l^2/\delta^2 \sim 10^8$, where l is the distance between mosaic boundaries.

Shaler and his collaborators have followed Frenkel's ideas and performed experiments which throw some light on the flow of metals at very high temperatures. Shaler concluded from his experiments that copper has a Newtonian viscosity small enough to explain sintering: the present authors conclude from the same experiments that the sintering of copper cannot be explained in terms of viscous flow.

The Apparent Viscosity of Copper at 850° C. (in poise)

Shaler and Wulff from sintering†	2.2×10^9
Udin <i>et al.</i> from creep ‡	7.8×10^{13}
Nabarro's theory of vacant lattice site diffusion	3.4×10^{12}

The argument may be put qualitatively in the following way. If copper behaves as a viscous liquid with surface tension, then, provided their radii are equal, the rate at which a pore in a porous mass closes should not be expected to differ by more than a factor 10 from the rate at which an isolated pore closes, or the rate at which an unloaded wire contracts. But the experiments show that the radii of pores in a powder compact decrease by 20% in 45 minutes at 850° C., while the radii of isolated pores showed no systematic decrease after heating for

* According to Kaye and Laby (1944) the viscosity of Lyle's golden syrup is 1.4×10^3 poise at 12° C.

† The value given is derived from data read from Shaler and Wulff's curves and is calculated by using the theory of § 5 (iii). Shaler and Wulff derive a value 5×10^8 sec/cm²; the dimensions of the poise are gm/cm/sec.

‡ Udin uses a formula due to Frenkel which is in error by a factor 2. His results should be expressed in the form $\eta = 260 \exp 59,000/RT$.

48 hours at 1,000° c., and the length of a free wire only decreased by 0.4% per day at 1,050° c.

The argument can also be expressed quantitatively by estimating the apparent viscosities for the various experiments. This is done in the Table, where the values are compared with the order of magnitude expected on Nabarro's theory; no estimate can be made in the case of the single-pore experiments, but the results are not inconsistent with those derived from Udin's wire-pulling experiment. The sintering experiment gives a value 10^4 times smaller than the wire-pulling experiment, and it is evident that the flow observed by Udin does not contribute appreciably to sintering; it probably arises from the Nabarro mechanism.

(iii) *The Mechanical Properties of Copper at High Temperatures*

No experiments have been made with the primary object of determining the deformation of metal specimens as a function of time and stress at sintering temperatures. It is therefore necessary to deduce the mechanical properties at these temperatures from more or less indirect evidence.

We have already seen that at the low stresses used in the wire-pulling experiments copper probably has a Newtonian viscosity. On the other hand, the sintering experiments cannot be so explained; for these the effective stress is somewhat higher.

It appears that no other experiments have been made on the mechanical properties of metals at high temperatures. However, Chalmers (1936) has measured the mechanical properties of tin at room temperature (212° c. below the melting point). He finds that for low stresses viscous flow occurs, the viscosity being 10^{13} poise, but that when the maximum shear stress exceeded a critical value of 10^7 dynes/cm² a vastly increased rate of deformation occurred, viz. a linear rate of strain of 15% per day at a shear stress of 1.4×10^7 dynes/cm².

The authors suggest that the mechanical properties of copper at high temperatures are similar to those found for tin, and that the important mechanism for deformation during sintering occurs when the yield stress is exceeded; Udin did not observe the yield stress because his stresses were too low. It will be shown that this view can explain the three experiments described in § 4(i), and in § 5(vi) it will be shown that the critical shear stress for copper at 850° c. should be at least 2.5×10^5 dynes/cm² or possibly greater by a factor less than 5. At room temperature the critical shear stress is 10^7 dynes/cm².

It is convenient at this point to formulate mathematically the two equations of mechanical state which will be used in the subsequent mathematical treatment. For a Newtonian viscous solid such as a glass the rate of shear strain $\dot{s}$ is proportional to the shear stress τ :

$$\tau = \eta \dot{s}, \quad \dots\dots(1)$$

where η is the coefficient of viscosity. This is represented in Figure 1 by the straight line through the origin. For a Bingham solid (Burgers 1939), which is appropriate to a crystalline metal,

$$\left. \begin{array}{l} \tau = \eta \dot{s} + \tau_c, \quad |\tau| > |\tau_c|, \\ \text{and} \quad \dot{s} = 0, \quad |\tau| < |\tau_c|, \end{array} \right\} \quad \dots\dots(2)$$

where τ_c is the critical shear stress and changes sign so as always to have the same sign as $\dot{s}$ (see Figure 1); for simplicity anisotropy of plastic properties will be

neglected. This relation between τ and $\dot{\epsilon}$ is the simplest which exhibits a yield point, and the instantaneous viscosity $\eta = \tau/\dot{\epsilon}$ decreases as the speed of deformation increases.

(iv) *Interpretation of Experiments on the Bingham Solid Model*

What really has to be explained is the qualitatively different behaviour of isolated pores and pores in the neighbourhood of other pores. In succeeding sections the problem is developed mathematically; here only a qualitative account is given of how this difference is explained by the hypothesis that a yield point exists.

In the neighbourhood of an isolated pore the shear stress caused by surface tension decreases with distance from the centre of the pore. Thus beyond a

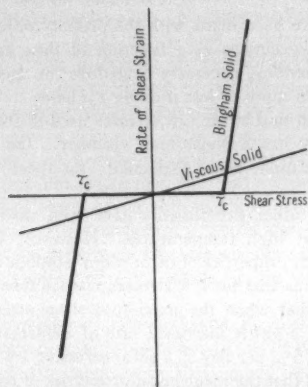


Figure 1. The relation between the rate of shear strain and the shear stress for a viscous solid and a Bingham solid.

certain distance the yield stress is not exceeded, and no flow can occur because the material beyond cannot flow. On the other hand, in a porous compact the plastic regions surrounding each pore will overlap, and flow will occur because the yield stress is exceeded everywhere; the interaction between pores is vital.

§ 5. THE THEORY OF SINTERING

In accordance with the discussion given in § 3, it will be assumed that deformation during sintering is due to surface tension and that the material of the particles has homogeneous mechanical properties defined by an equation of state connecting the rate of (octahedral) shear strain with the (octahedral) shear stress. The material will also be assumed to be incompressible. Further, in order to arrive at definite quantitative results and to simplify the calculations, it has been convenient to consider a special model in which all the pores are isolated equal spheres distributed at random in the real solid material. This assumption is considered at the end of sub-section (i) and does not essentially change the character of the results.

In the next sub-section results independent of the form of the equation of state are deduced and the general nature of the flow during sintering is indicated;

the model used for making detailed calculations is described and the general equation of energy for the flow is set up. Two equations of state, that for a viscous solid and that for a Bingham solid, are considered in detail in sub-sections (iii) and (iv), and equations giving the density as a function of the time are derived. In sub-section (v) the effect of an initial viscosity is considered. Finally, in sub-section (vi) the results are compared with the experiments described in §4 (i) and an estimate made of the critical shear stress for copper.

(i) *Conclusions Independent of the Form of the Equation of State*

Consider a compact in which all the pores are spherical and of equal radius r_1 . The surface tension γ acting in the surfaces of the pores is equivalent to a pressure $-2\gamma/r_1$ inside all the pores; this produces shear stresses in the material and the pores close. Now because the material is supposed incompressible, the application of an additional hydrostatic pressure to the inside of all the pores and to the external surfaces of the compact will cause no strain and hence no change in the rate of strain. If this additional pressure is $+2\gamma/r_1$, the resultant pressure will be zero inside all the pores and $+2\gamma/r_1$ on the external surfaces. Thus the effect of surface tension in closing the pores is equivalent to the application of an external pressure to the surface of the compact. Incompressibility is not essential to this result, it is sufficient that the rate of strain be independent of the hydrostatic pressure.

If attention is directed not to the detailed nature of the flow in the immediate vicinity of a pore but to the behaviour of volume elements that contain a large number of pores, then the compact can be regarded as forming a homogeneous continuum. When a hydrostatic pressure is applied to the surface of the continuum, its volume slowly decreases; the volume of the real material remains unchanged, but because of shear stresses its shape changes and it flows into the pores. Now the application of a hydrostatic pressure to the surface of a homogeneous continuum produces an equal uniform hydrostatic pressure at all points throughout the whole volume, independently of its shape or size (Love 1944). Thus the rate of increase in density of any element of volume will be uniform throughout the whole body and the rate of sintering will be independent of the shape or size of the compact (cf. §2). A consequence of this uniform contraction is that to an observer situated on a pore, and moving with it, all distant pores will appear to approach with a velocity proportional to their distance (see Eddington 1933). During this flow the size of the pores will decrease, their centres will approach each other, but the total number of pores will remain unchanged.

Even when the pores are not equal in size and are not spherical the rate of increase in density will be uniform throughout the whole compact, provided the distribution of pore shape and size is the same in all parts of the compact. For if a mean pressure $2\gamma/\bar{r}_1$ is applied to all surfaces of the real material, where $\bar{r}_1$ is some mean radius of curvature of all the pore surfaces, then the resultant forces on the real material are a pressure $2\gamma/\bar{r}_1$ over the external surface of the compact and a distribution of pressure over the surfaces of the pores the effect of which is zero on the average. This problem is discussed in more detail by Mackenzie (1950). When the pores are not spherical flow will contribute to their sphering. When the pores are not equal in size the small pores will shrink and disappear more rapidly than do the larger pores, so that the total number of pores will decrease with time of sintering.

(ii) *The Model*

The problem is to deduce the properties of the equivalent homogeneous material from the properties of the real material and the number and the size of the pores. A convenient self-consistent method of approximation has been indicated by Fröhlich and Sack (1946); their method is equivalent to a perturbation calculation and is valid when the volume of the pores is sufficiently small compared with the volume of the material. This method is used in the following calculations.

Every pore of radius r_1 will be surrounded by a spherical shell of the real incompressible material, out to some radius r_2 which will be chosen later (Figure 2). When an external pressure is applied there will be certain stresses

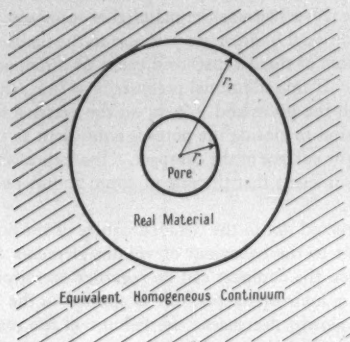


Figure 2. The model. The reaction of material outside the sphere of radius r_2 is calculated by replacing it by an equivalent homogeneous continuum.

acting across the outer boundary of this shell; these stresses represent the combined effect of the external pressure and the interaction of the rest of the porous medium. If these stresses were known, the flow of the material inside the shell could be calculated and the decrease in volume determined; then by summation for all pores the total decrease in volume is known since the real material is incompressible. The approximation consists in replacing the material outside the spherical shell (including the pores it contains) by an equivalent continuum. For this approximation to be consistent it is necessary that both the macroscopic flow and the density of the equivalent continuum should not be altered by the presence of the pore and its surrounding shell.

Now the stress in the continuum is everywhere a hydrostatic pressure equal to the applied pressure. Thus the first condition of consistency implies that the stress at the outer surface of the spherical shell containing the pore is a pressure equal to the applied pressure, or the rate of increase of density of the continuum is the same as that of the pore and its surrounding shell when the same pressure is applied to its outer surface. The second condition of consistency determines r_2 ; if ρ is the density of the compact relative to that of the real material, then the condition that the pore and its shell have the same density as the compact is

$$\rho = 1 - (r_1/r_2)^3. \quad \dots\dots(3)$$

The approximations of this treatment are valid when r_1/r_2 is small compared with unity, and it is not clear how accurate the treatment will be when the pores are close together. However, the results are correct as ρ tends to zero, so that the method should give the correct order of magnitude for all densities (Mackenzie 1950).

The problem has now been reduced to the calculation of the rate of decrease in radius r_1 of a spherical pore, surrounded by a shell of incompressible but shearable material, when a pressure $-2\gamma/r_1$ is applied inside the pore. In the case of an isolated pore the outer radius r_2 would be infinite; the finite value for r_2 takes into account the interaction between the pores. The rate of closing will be calculated by equating the energy dissipated by the flow of the material in the shell to the work done by surface tension; the kinetic energy of the material will be negligibly small, so that it is possible to treat the problem as one of a steady state.

Consider a regular octahedron in which the line joining a pair of opposite vertices is directed along a radius of a pore. Then when a hydrostatic pressure is applied inside the pore the flow of material is radial and the faces of the octahedron are subject only to shear strain—the octahedral shear strain. For a Bingham solid the instantaneous viscosity η is a function of the shear strain. Since the latter is a tensor quantity, some combination of the components must be chosen on which η is to depend. By analogy with the theory of plasticity (Nádaí 1937) it will be assumed that at constant temperature the equation of state connects the rate of octahedral shear strain with the corresponding octahedral shear stress.

If the rate of radial strain is $\dot{\epsilon}$, the condition of incompressibility shows that the rate of strain in any direction perpendicular to the radius is $-\frac{1}{2}\dot{\epsilon}$; these are three principal directions of stress and strain. The corresponding octahedral shear strain is

$$\dot{s} = \sqrt{2} \dot{\epsilon}. \quad \dots\dots (4)$$

The rate of dissipation of energy in any element can be found from the theory of viscosity as given, for example, by Lamb (1932). In Lamb's notation $a = \dot{\epsilon}$, $b = c = -\frac{1}{2}\dot{\epsilon}$, $f = g = h = 0$, and the rate of dissipation of energy per unit volume* is

$$\dot{E} = 3\dot{\epsilon}^2 \eta. \quad \dots\dots (5)$$

The viscosity may be a function of position. The corresponding results for a cylindrical pore are $\dot{s} = 2\dot{\epsilon}$ and $\dot{E} = 4\dot{\epsilon}^2 \eta$. The relations have been obtained by Eshelby (1949) by elementary means.

If u_1 is the radial velocity of the surface of the pore, surface tension does work at a rate $-4\pi r_1^2 u_1 2\gamma/r_1$, and on equating this to the energy dissipated throughout the whole volume of the shell

$$-2\gamma r_1 u_1 = 3 \int_{r_1}^{r_2} r^2 \dot{\epsilon}^2 \eta(\dot{\epsilon}) dr. \quad \dots\dots (6)$$

Now, since the real material is incompressible, the radial velocity at any radius is inversely proportional to the square of the radius. Thus $u = u_1 r_1^2/r^2$ and the rate of radial strain

$$\dot{\epsilon} = \frac{du}{dr} = -\frac{2u_1 r_1^2}{r^3}. \quad \dots\dots (7)$$

* Frenkel (1945), treating the same problem, assumes wrongly that $b=c=0$, so that his result is two-thirds of ours.

When $\eta(\epsilon)$ is given explicitly, (6) and (7) can be solved for u_1 and, by means of (3), a differential equation for ρ as a function of time derived. This is done in the following sub-sections for $\eta = \text{constant}$ (viscous solid) and $\eta = \eta_\infty + \tau_c/\dot{s}$ (Bingham solid).

(iii) *The Sintering of a Viscous Solid*

For a solid which has a Newtonian viscosity, η is independent of the rate of strain. On substituting (7) into (6) and making use of (3) it is found that

$$u_1 = -\frac{\gamma}{2\eta} \frac{1}{\rho} \quad \dots\dots(8)$$

For an isolated pore $\rho = 1$, so the factor $1/\rho$ represents the effect all the other pores have on the rate of closing of one pore; since ρ will always be greater than $\frac{1}{2}$ in a powder compact, a pore in a compact will never close at more than double the rate of an isolated pore. Equation (8) is also appropriate to the closing of a cylindrical pore.

In experiments on sintering, the compact is usually held at constant temperature and the density measured as a function of time. The volume of real material in the compact does not change, nor does the total number of pores, if they are all equal. It is therefore convenient to obtain a relation between the relative density and the time of sintering in terms of n , the number of pores per unit volume of the real material, and γ and η . The volume of each pore is $4\pi r_1^3/3$, so that equation (3) gives

$$r_1 = \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \cdot \frac{(1-\rho)^{\frac{1}{3}}}{\rho^{\frac{1}{3}}} \cdot \frac{1}{n^{\frac{1}{3}}}, \quad \dots\dots(9)$$

and, since $u_1 = dr_1/dt$,

$$\frac{d\rho}{dt} = \frac{3}{2} \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \frac{\gamma n^{\frac{1}{3}}}{\eta} \cdot (1-\rho)^{\frac{1}{3}} \rho^{\frac{1}{3}}.$$

When this is integrated the time is obtained as a function of ρ ,

$$\frac{\gamma n^{\frac{1}{3}}}{\eta} (t - t_0) = \frac{2}{3} \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \int_0^e \frac{d\rho}{(1-\rho)^{\frac{1}{3}} \rho^{\frac{1}{3}}}, \quad \dots\dots(10)$$

and ρ is then determined as a function of t by inversion. The indefinite integral is easily evaluated by means of the substitution $x^3 = (1-\rho)/\rho$ and the result is

$$\frac{1}{2} \ln \frac{1+x^3}{(1+x)^3} - \sqrt{3} \tan^{-1} \frac{2x-1}{\sqrt{3}}.$$

In Figure 3 the curve $a=0$ shows ρ as a function of the reduced time $\gamma n^{\frac{1}{3}}(t-t_0)/\eta$. The zero from which the time of sintering of any particular compact is measured is determined by the initial density, and in the Figure $t=t_0$ is chosen, arbitrarily, to correspond to $\rho=0$. An increase in the number of pores per unit volume of material makes the abscissa scale correspond to a shorter time but does not change the curve. It is clear that sintering to unit relative density occurs in a finite time. If the reduced time corresponding to a given density is read off the curve and plotted as a function of $(t-t_0)$ found experimentally, a straight line of slope $\gamma n^{\frac{1}{3}}/\eta$ will be obtained; n can be estimated from the size and packing of the particles and so γ/η can be deduced.

Shuttleworth (1949a) has derived equation (10) by an alternative method involving the same assumptions. The problem of the steady flow of a viscous liquid is mathematically identical with the determination of the elastic displacement

in an incompressible solid, and so it is possible to solve the viscous problem by means of the known solution to the elastic problem. The problem is also discussed in more detail by Mackenzie (1950).

(iv) *The Sintering of a Bingham Solid*

The instantaneous viscosity of a Bingham solid is found from (2) and (4) to be

$$\eta = \eta_{\infty} + \tau_0 / \sqrt{2\dot{\epsilon}},$$

and if flow occurs at all, τ_0 and $\dot{\epsilon}$ have the same sign. Substituting this value of η into (6) and using (7), it is found that

$$-u_1 = \frac{\gamma}{2\eta_{\infty}\rho} \left[1 - \frac{3\sqrt{2}\tau_0 r_1}{2\gamma} \ln \frac{r_2}{r_1} \right], \quad \dots\dots(11)$$

and $\dot{\epsilon}$ or $-u_1$ has the same sign as τ_0 only if

$$2\gamma/r_1 > 3\sqrt{2}\tau_0 \ln(r_2/r_1). \quad \dots\dots(12)$$

A pore will not close unless this condition is satisfied.

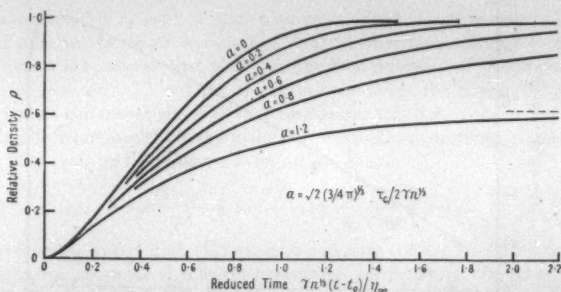


Figure 3. The relative density of a compact as a function of the reduced time when the real material of the compact behaves like a Bingham solid. The curve $\alpha=0$ applies when the real material behaves like a viscous solid.

Equation (12) may be written in the form

$$2\gamma/r_1 > \sqrt{2}\tau_0 \ln\{1/(1-\rho)\}. \quad \dots\dots(12a)$$

Thus for a given initial density there is a critical size of pore such that a material containing smaller pores will increase in density while one containing larger pores will not.

A pore in a Bingham solid will behave quite differently according to whether it is near or far from other pores. An isolated pore will not close since r_2/r_1 is large, whilst in a compact the interaction between the stress fields of the pores is vital and sintering will occur because of the small value of the ratio r_2/r_1 . The corresponding criterion for closure of a cylindrical pore is

$$\gamma/r_1 > 2\tau_0 \ln(r_2/r_1). \quad \dots\dots(13)$$

This criterion is applicable to Shaler's experiments on isolated cylindrical pores because the pressure due to surface tension in the outer surface (including the ends) is small compared to pressure at the inner surface.

Using equations (3) and (9), the rate of increase of density of a compact is given by

$$\frac{d\rho}{dt} = \frac{3}{2} \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} \cdot \frac{\gamma n^{\frac{1}{3}}}{\eta_{\infty}} \cdot (1-\rho)^{\frac{2}{3}} \rho^{\frac{1}{3}} [1 - a(1/\rho - 1)^{\frac{1}{3}} \ln \{1/(1-\rho)\}], \quad \dots (14)$$

where

$$a = \sqrt{2} \left(\frac{3}{4\pi} \right)^{\frac{1}{3}} \cdot \frac{\tau_c}{2\gamma n^{\frac{1}{3}}}.$$

This equation was integrated numerically and, in Figure 3, ρ is plotted as a function of the reduced time $\gamma n^{\frac{1}{3}}(t - t_0)/\eta_{\infty}$ for a number of values of the parameter a . The value $a = 0$ applies to a viscous liquid, and increasing values of a correspond to increasing values of the yield stress τ_c or to decreasing values of n .

In Figure 4 the function

$$f(\rho) = (1/\rho - 1)^{\frac{1}{3}} \ln \{1/(1-\rho)\}$$

is plotted as a function of ρ . From equation (14) it is clear that sintering occurs only when $af(\rho)$ is less than unity. When a is small this condition is satisfied for all ρ and sintering to unit relative density always occurs in a finite time. When

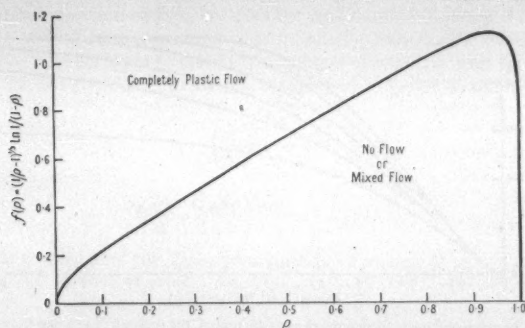


Figure 4. The function $f(\rho)$ which determines the character of the flow in a compact.

a is greater than 0.89 the rate of sintering decreases to zero as the density increases towards a value such that $f(\rho) = 1/a$; if the initial relative density is less than 0.94 this limiting relative density is approached asymptotically. If, however, with the same value of a , the initial relative density is increased (e.g. by pressing), then at some value greater than 0.94 sintering will again begin and the compact will sinter to unit relative density in a finite time. This second branch of the curve begins at $\rho = 0.998$ for $a = 1.2$, and is not shown in the Figure. The fact that $f(\rho)$ tends rapidly to zero as ρ tends to unity indicates that very small pores will close even when they are at large distances apart.

(v) The Effect of an Initial Viscosity

It was assumed in the last sub-section that when the shear stress was less than τ_c no movement occurred. This is not strictly true, and in this sub-section the effect of an initial viscosity is considered. It will now be supposed that for $\tau < \tau_c$ the rate of shear strain is proportional to the shear stress, the constant of proportionality being $1/\eta_0$, corresponding to an initial coefficient of viscosity η_0 ,

while for $\tau > \tau_c$ the slope of the line defining the equation of state is assumed to be $1/\eta_\infty$ as for the Bingham solid just considered. The initial viscosity η_0 will be at least ten thousand times larger than η_∞ for copper.

If the critical shear stress is nowhere exceeded, i.e. $s_1 < \tau_c/\eta_0$, the flow will be viscous everywhere and the equations of § 5 (iii) will apply; however, there will usually be a transition boundary inside which the material is flowing plastically, and outside which the material is flowing viscously. At this boundary (of radius r_c) $\dot{s} = \tau_c/\eta_0$ and equations (4) and (7) give

$$\frac{r_1^3}{r_c^3} = \frac{-\tau_c r_1}{2\sqrt{2}u_1\eta_0}, \quad \dots\dots(15)$$

which determines the radius of the boundary. If $r_c \leq r_1$ the flow will be viscous everywhere and equation (8) shows that this will be so for $r_1 > \sqrt{2\gamma/\tau_c\rho}$; if $r_c \geq r_2$ the flow will be plastic everywhere and the equations of the last sub-section apply. On the other hand, if $r_1 \leq r_c \leq r_2$ there is a transition boundary and (15) together with (3) and (9) show that

$$-u_1 \leq \frac{\gamma}{2\eta_0\rho} \frac{a\rho^{\frac{1}{2}}}{(1-\rho)^{\frac{1}{2}}}, \quad \dots\dots(16)$$

where $a = \sqrt{2(3/4\pi)^{\frac{1}{2}}\tau_c/2\gamma n^{\frac{1}{2}}}$ is the parameter defined previously. Practical values of a lie in the neighbourhood of unity, so that for $\rho < 0.99$ a comparison with equation (11) shows that the velocity when any part of a compact flows viscously is less than 1% of the initial velocity when the flow is fully plastic. The neglect of the initial viscosity therefore seems justified.

Using the equation of energy as in the previous sub-sections, the condition that the flow shall be fully plastic is found to be

$$1 - a \frac{(1-\rho)^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} \left[\ln \frac{1}{(1-\rho)} + \frac{\eta_\infty}{\eta_0} \frac{1}{1-\rho} \right] \geq 0. \quad \dots\dots(17)$$

This corresponds to equation (12) expressed in terms of ρ . The additional term involving η_0 has the same sign as the logarithmic term, so that the curve defining the transition to mixed flow lies very slightly above the curve drawn in Figure 4, except for values of ρ very close to unity, where the new curve rises to infinity. This rise simply means that when ρ is very close to unity the flow at the outer boundary always becomes slow enough for an outer region to be viscous. Thus the previous discussion of Figure 4 applies unchanged except that there is always mixed flow ultimately, and sometimes it can become plastic again before the final mixed flow. Further, the barrier can always be penetrated slowly since the flow is partly viscous in this region.

(vi) Comparison of Theory with Experiment

It is to be expected that the viscous solid model will explain the sintering of glass, but there appear to be no data on the increase in density of glass compacts as a function of the time of sintering.

It is possible to explain the experiments of Shaler and Udin discussed in § 4 (i) if it is supposed that at high temperatures copper behaves like a Bingham solid, and a rough estimate can be made of the yield stress τ_c by using the data for these experiments.

In Udin's wire-pulling experiments the greatest simple tension that was applied to the wires of 36μ radius was 8.1×10^5 dynes/cm²; when corrected

for the surface tension stress this corresponds to an octahedral shear stress of 2.0×10^5 dynes/cm². At this stress yield has not occurred and, therefore, $\tau_e > 2.0 \times 10^5$ dynes/cm² at 1,000° c.

Rapid sintering occurred in Shaler's experiments at 850° c., so that the yield stress must have been exceeded. For $\gamma = 1,500$ dynes/cm, equation (12) leads to $\tau_e < 2.6 \times 10^5$ dynes/cm². Since the experiments lasted for only a short time, and no large increase in density occurred, the pores would not be spherical. This would increase the pressure due to surface tension by some factor which is probably less than 5. Therefore the yield stress at 850° c. might well be greater than 2.6×10^5 dynes/cm².

Shaler found that cylindrical pores drilled in a copper wire did not close when heated to 1,000° c. It follows from equation (13) that $\tau_e > 0.2 \times 10^5$ dynes/cm²; in fact, if $\tau_e > 2 \times 10^5$ dynes/cm² the whole of the wire must have been flowing viscously.

At room temperature the critical shear stress of copper single crystals is 10^7 dynes/cm².

§6. THE INFLUENCE OF GAS ON SINTERING

Metals are often sintered in atmospheres of inert gases, and even when the sintering is carried out in a vacuum the pores, once closed, will contain gases evolved from the metal. Sometimes decomposable carbonates are deliberately added to glasses before sintering in order to obtain low-density products with bubble structures.

Gases can affect sintering in two ways (a) by their chemical effect on the surface of the powder, or (b) by their hydrostatic pressure. Jordan and Duwez (1949) find that copper sinters more quickly in hydrogen than in a vacuum. This is probably due to the reduction of copper oxide on the surface of the powder by the hydrogen. The changes that occur when metal surfaces are heated in inert gases have recently been reviewed by Shuttleworth (1948).

In this paper only the hydrostatic effects of gas pressure will be discussed. The pressure due to surface tension inside a pore of 30μ radius is about one atmosphere in the case of copper, so that even a small gas pressure inside the pores will reduce the rate of sintering and may even cause the pores to expand. If p is the excess of the gas pressure at the surface of the compact over that inside the pores, the rate of sintering when the pores are equal spheres is obtained by replacing $2\gamma/r$ by $2\gamma/r + p$ in the expressions for u_1 and dp/dt . The integrals have only been evaluated for the case of a viscous solid, but the results are qualitatively the same whatever form the equation of mechanical state takes.

The gas pressure inside a pore will vary in a manner that depends on the amount of gas initially present in the pore, the size of the pore and the rate at which it closes. Two cases will be considered in detail: (a) a constant pressure difference between the pores and the surface of the compact, and (b) a constant mass of gas in each of the pores but no pressure outside the compact.

(i) Constant Pressure inside the Pores

If large quantities of gas can dissolve in the metal and the rate of diffusion is large enough to ensure that equilibrium is always maintained between the gas in the pores and that in the metal, or if a reversible chemical reaction involving the gas occurs at the surface of the pores, then the gas pressure inside the pores will remain constant when the size of the pore changes. A constant pressure

difference also occurs when a constant pressure is applied to the surface of the compact and there is no gas pressure inside the pores.

For a viscous solid γ must be replaced by

$$\gamma(1 + pr_1/2\gamma) = \gamma\{1 + b(1 - \rho)^{1/2}/\rho^{1/2}\},$$

where $b = (3/4\pi)^{1/2}(p/2\gamma n_1)$, so that instead of equation (10)

$$\frac{\gamma n^3}{\eta}(t - t_0) = \frac{2}{3} \left(\frac{3}{4\pi}\right)^{1/2} \int_1^{\rho} \frac{d\rho}{(1 - \rho)^{3/2} \rho^3 [1 + b(1 - \rho)^{1/2}/\rho^{1/2}]} \quad \dots (18)$$

After the substitution $x^3 = (1 - \rho)/\rho$ the indefinite integral is

$$\frac{2b+1}{b^2+b+1} \frac{1}{2} \ln \frac{1+x^3}{(1+x)^3} - \frac{\sqrt{3} \tan^{-1}(2x-1)/\sqrt{3}}{b^2+b+1} - \frac{3b^2}{b^3-1} \ln \frac{1+bx}{1+x}.$$

In Figure 5, ρ is shown as a function of the reduced time, $\gamma n^3(t - t_0)/\eta$, for a

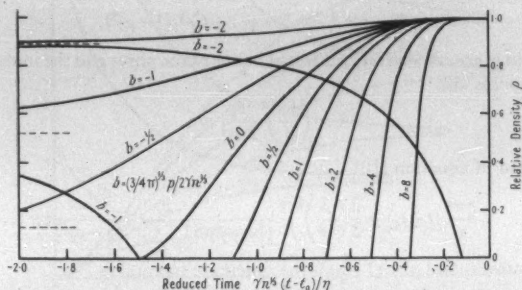


Figure 5. The relative density of a compact as a function of the reduced time when various constant pressures are applied inside the pore, or outside the compact, and the real material behaves like a viscous solid.

number of values of the parameter b . The zero of time has been chosen so that $t = t_0$ when $\rho = 1$; when the curves are used to predict the behaviour of a compact the zero is decided by the initial density of the compact. The parameter b is proportional to the pressure difference p between the pores and the outside of the compact; when $b = 1$, $n^3 = 100 \text{ cm}^{-1}$ and $\gamma = 1,500 \text{ dynes/cm.}$, $p = 0.5$ atmosphere. Positive b corresponds to an externally applied pressure and negative b to a pressure inside the pores.

For b positive the compact sinters to unit density in a finite time and the application of a few atmospheres pressure to the outside of the compact reduces the time of sintering to about one-quarter of that required when no pressure is applied. An even more rapid increase in the rate of sintering might be expected if a few atmospheres pressure were applied to the outside of a compact of a material for which the equation of state is similar to that for a Bingham solid. In particular, if the parameter a is large it should be possible to prevent the increase of density from stopping at some finite value less than the theoretical density.

When b is negative there is a critical density at which the pressure of the gas inside the pores just balances the surface tension. This occurs when $-p = 2\gamma/r_1$, and the corresponding critical density can be calculated from equation (9). When the initial density is greater than the critical density the pressure due to surface tension always overbalances the gas pressure, and sintering to unit relative density occurs in a finite time. On the other hand, when the initial density is

less than the critical density the pores expand and continue to do so at an ever increasing rate. However, the density will never decrease as rapidly as shown in Figure 5: for the expansion will be controlled by the rate at which gas can diffuse into the pores and the later stages of the expansion will take place more as though there were a constant mass of gas in the pores.

(ii) *Constant Mass of Gas inside each Pore*

If an inert gas such as nitrogen is trapped in a compact during pressing it will not dissolve or diffuse appreciably during sintering and the mass of gas inside each pore will remain constant. The pressure exerted by such gas will vary inversely as the volume of the pore, and always opposes the pressure due to surface tension (p is always negative).

For a viscous solid γ must be replaced by

$$\gamma(1 + p_0 r_0^3 / 2\gamma r_1^2) = \gamma[1 + C\rho^{\frac{1}{3}}/(1-\rho)^{\frac{2}{3}}],$$

where r_0 and p_0 are respectively the initial radius of the pores and the initial pressure inside the pores and

$$C = \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \frac{p_0}{2\gamma n^{\frac{1}{3}}} n r_0^3.$$

Then instead of equation (10)

$$\frac{\gamma n^{\frac{1}{3}}}{\eta} (t - t_0) = \frac{2}{3} \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \int_0^{\rho} \frac{d\rho}{(1-\rho)^{\frac{1}{3}} \rho^{\frac{1}{3}} [1 + C\rho^{\frac{1}{3}}/(1-\rho)^{\frac{2}{3}}]}. \quad \dots (19)$$

After the substitution $x^3 = (1-\rho)/\rho$ the indefinite integral is

$$\begin{aligned} & -\frac{C-1}{C^3+1} \frac{1}{2} \ln \frac{1+x^3}{(1+x)^3} - \frac{C+1}{C^3+1} \sqrt{3} \tan^{-1}(2x-1)/\sqrt{3} \\ & + \frac{C^2}{C^3+1} \frac{1}{2} \ln \frac{x^2+C}{(1+x^3)^2} - \frac{3(-C)^{\frac{1}{3}}}{C^3+1} \ln \frac{x-(-C)^{\frac{1}{3}}}{x+(-C)^{\frac{1}{3}}} \end{aligned}$$

for C negative.

In Figure 6, ρ is shown as a function of the reduced time, $\gamma n^{\frac{1}{3}}(t-t_0)/\eta$ for a number of values of the parameter C ; the zero of time has been chosen so that $t=t_0$ when $\rho=0$. When these curves are used to predict the behaviour of a particular compact, the value of C is determined by the initial pore radius and initial gas pressure, and the zero of time is specified by the initial density. The parameter C is proportional to the initial pressure inside the pores; when $C=-1$, $n^{\frac{1}{3}}=1/r_0=100 \text{ cm}^{-1}$ and $\gamma=1,500 \text{ dynes/cm.}$, $-p_0=0.1 \text{ atmosphere.}$

Corresponding to any initial mass of gas inside each pore, there is an equilibrium pore radius at which the gas pressure just balances the pressure due to surface tension; this occurs for $r_1^3 = -p_0 r_0^3 / 2\gamma$, and the equilibrium density is found from equation (9). The density of a compact approaches the equilibrium density asymptotically; it decreases or increases according as the initial density is greater or less than the equilibrium density.

It has been seen that very small gas pressures can have quite large effects. When the pores are not all equal in size, or the mass of gas in each is not proportional to the initial pore volume, then the behaviour of any individual pore may be anomalous. In particular, suppose that some soluble gas such as hydrogen exerts a constant pressure p inside all pores and that the pores are of various sizes,

then the larger pores for which $2\gamma/r < p$ will expand while the smaller pores will contract. Thus great care is needed in interpreting experimental results concerning the change in size distribution of pores during sintering.

§ 7. ATOMIC MECHANISMS IN SINTERING

A complete description of the atomic processes by which the deformation of metals occurs during sintering must await further experiments. However, the speed with which deformation occurs and the presence of a yield point indicates that a dislocation mechanism is probably operative. The yield point may possibly be due to dislocations breaking away from atmospheres of solute atoms which lock them (Cottrell 1948, Cottrell and Bilby 1949); this would imply a temperature-sensitive yield point.

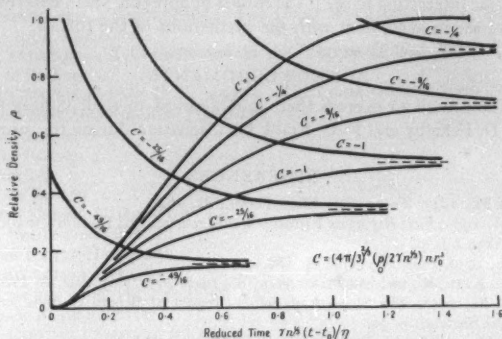


Figure 6. The relative density of a compact as a function of the reduced time when there is a constant mass of gas inside each pore and the real material behaves like a viscous solid.

The fact that deformation is caused by the change in surface energy puts a restriction on the possible modes of deformation. Deformation cannot occur by a series of catastrophic processes in each of which one block of the crystal is rapidly displaced with respect to another, since this would increase the surface area by the formation of slip lines. In order that appreciable rumpling of the surface shall not occur, the rate of deformation must be slow compared with the rate at which rearrangement of the surface atoms can take place by means of surface and volume diffusion. Hansen (1939) has shown that slip lines do not appear on the surface of aluminium when it creeps at high temperatures.

Shuttleworth (1949b) has discussed the distinction between the surface free energy and the surface tension of solids. These are equal only when the surface energy is independent of the strain, a condition which does not, in general, hold for solids.

§ 8. SUGGESTED EXPERIMENTS

The theory developed in this paper connects the rate of increase of density during sintering with the mechanical properties of the bulk material. There are no adequate experimental data on either of these aspects of the theory, and some experiments which could usefully be carried out are suggested in the following paragraphs.

There is a need for experiments which determine the laws of deformation of materials at sintering temperatures and under small stresses of a few atmospheres

or less. The rate of shear strain should be determined as a function of the shear stress. This would decide whether or not metals have a yield point at low stresses, and whether or not there exists a simple equation of mechanical state in which the rate of shear strain depends only on the shear stress and the temperature. These experiments should also be of importance for suggesting the mechanisms by which deformation takes place at sintering temperatures.

There is also a need for experiments which determine the density of a powder compact as a function of the pore size, the time and the temperature. These experiments could most usefully be done with glass which behaves as a homogeneous viscous solid, and with silver for which there would be no complications due to oxide films. The experiments should be carried out in a vacuum and the particles used should be equal spheres and thoroughly degassed.

It would be interesting to see if the results of applying small external pressures to a compact are in agreement with the predictions of the theory.

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ABSTRACTS FOR SECTION A

Radiative Reaction and Damping in Scattering, by J. HAMILTON.

ABSTRACT. The higher order radiative corrections and the damping corrections to the cross section for the scattering of an electron by an electrostatic potential are investigated. It is shown that the perturbation treatment of the radiative corrections at high frequencies joins on to the Bloch-Nordsieck solution for low frequencies. The damping effect arising from the low frequency photons is shown to be negligible, and a modified damping equation is proposed which gives a smooth transition to the damping correction in passing from high to low frequency photons.

Directional Correlation between Successive Internal-Conversion Electrons, by J. W. GARDNER.

ABSTRACT. For the internal conversion of successive electric multipole quanta by s-electrons, unaccompanied by spin reversal, the directional correlation between the ejected electrons can be derived by symmetry arguments alone. A general formula is established and special cases likely to be of practical interest are tabulated. Comparison is made with Hamilton's results for γ - γ correlation.

Applications of the Peierls-McMannus Classical Finite Electron Theory, by J. IRVING.

ABSTRACT. In the case of an electron in a high-frequency field of small amplitude it is found that the scattering cross section for radiation depends only on the mechanical mass m , instead of the total mass, as in the point electron theory. For the scattering of fast electrons by "protons" the differential cross section for scattering depends on the mass in a more complicated way. Finite values are obtained for the deflection for all values of the impact parameter and for the total energy radiated in the collision.

Perturbation Theory in Neutron Multiplication Problems, by K. FUCHS.

ABSTRACT. The perturbation theory for neutron diffusion problems in the one group theory is developed systematically by expansion in the system of orthogonal solutions of the integral equation in Part I of the paper.

The multiplication of a neutron source inside a fissile system is discussed. The multiplication factor for a near critical system is derived as a function of the position of the source; isotropic sources are considered in Part II and anisotropic sources in Part III.

Part IV is concerned with a few practical applications of the theory.

Half-lives of ^{141}Ce and ^{169}Yb , by D. WALKER.

ABSTRACT. The half-lives of ^{141}Ce and ^{169}Yb , produced through slow neutron capture in cerium and ytterbium, have been redetermined as (33.11 ± 0.23) days and (31.83 ± 0.21) days respectively. The errors appear to be smaller than in previous determinations.

The Decay and Capture of μ -Mesons in Photographic Emulsions, by M. G. E. COSYNS, C. C. DILWORTH, G. P. S. OCCHIALINI, M. SCHOENBERG and N. PAGE.

ABSTRACT. Observations on the decay and capture of μ -mesons in Kodak NT4 and Ilford G5 photographic emulsions are described. It is found, in agreement with other workers, that $63 \pm 4\%$ of all μ -mesons stopping in the emulsion decay with the emission of a fast electron, a result which is shown to be in reasonable agreement with theoretical predictions. The emission of slow electrons is observed from $7.2 \pm 1.3\%$ of all slow mesons, and these are ascribed mainly to an Auger effect accompanying the capture of μ -mesons.

A Contribution to the Theory of Soft X-ray Emission Bands of Sodium, by P. T. LANDSBERG.

ABSTRACT. A wave-mechanical treatment of the $L_{II,III}$ emission spectrum of sodium is given, taking into account (i) the broadening of the energy levels in the conduction band due to electron collisions (§§ 2-4); (ii) the lowering of the potential energy of the electron about to make transition into an inner shell of an ion, due to electrostatic attraction (§§ 5-6). The first problem is treated in the free electron approximation, using a model suggested by Skinner, and the theoretical curve has a 'tail', as found experimentally. The calculation shows that a modification of the wave functions of electrons in a free electron model may be required, and this possibility is discussed. In the second problem, the depression in the electron potential is approximated by using a spherical potential well in the neighbourhood of the ion. The first effect proves to be the more important one, and there is good agreement with experiment.

The Luminescence Characteristics of Tin Activated Zinc Sulphide Phosphors, by G. F. J. GARLICK and D. E. MASON.

ABSTRACT. It has been found that the inclusion of stannous compounds in relatively large concentrations in the preparation of zinc sulphide phosphors results in an intense red luminescence when excitation is by ultra-violet light of long wavelength (3650-4000 Å.). Such characteristics as the luminescence spectra, excitation spectra, phosphorescence and thermoluminescence and the variation of luminescence with temperature have been studied. The hexagonal crystal form of zinc sulphide is essential to the production of efficient phosphors with tin as the activating impurity.

Obituary Notices

WILLIAM BOWEN

WILLIAM BOWEN, Managing Director of the Bowen Instrument Co. Leeds, died on 9th February 1949. He was born in September 1886, the eighth child and third son of John and Eliza Bowen. He had three brothers and seven sisters. His father was a coal miner; so was the eldest brother (9½ years older) and the second brother, David Bowen, K.C. (1½ years older).

He was educated at the Pentre Council Schools until the age of twelve. When he reached the school leaving age he wanted to become a coal miner but was persuaded to continue his education by entering the Porth County School; he remained there until, at the age of 17, he won a scholarship to the University College, Cardiff, where he took the courses in engineering. During that time (1904-5) the Welsh Revival movement occurred; with this Bowen was so taken up that he neglected his studies and lost his scholarship. His parents maintained him at College from 1905-6, but he did not succeed in getting his degree in engineering.

For a while after leaving college, in 1906, he taught at an elementary school in Swansea and then at a coaching school in London, after which he obtained a subordinate appointment with the Great Eastern Railway in the Engineers' Department. From there he entered the service of the Cambridge Instrument Company. Towards the end of 1913 he left this company to join his brother David Bowen as consulting engineers in Leeds. There he soon developed his own instrument company specializing in pyrometry, and up to the outbreak of the 1939 war his work was solely concerned with temperature measuring equipment.

In October 1939 he was asked to undertake the manufacture of compensating leads for aero engine temperature equipment. The insulation of these leads was of the plastics type and Bowen brought to bear his knowledge of temperature control equipment to secure a uniform coating of plastic on the cables. He then launched out into the production of plastic-covered wire and a vast amount was supplied in the form of assault cable for military purposes. He also produced a variety of plastic goods.

In the post war years he made strenuous efforts to popularize plastic-covered cables and read a paper on the subject before the Institution of Electrical Engineers. During this period he employed scientific assistance in development work. He contributed funds to the Scientific Instrument Manufacturers Association for the establishment of an award known as 'The Bowen Prize'. The resumption of peace time conditions and business competition imposed a severe strain which his constitution could not stand.

In 1946 Bowen and the writer made a B.I.O.S. trip to Germany and a warm friendship was formed as the result of it. He was a man of charming personality and was highly respected by his employees whose welfare was his first consideration.

In reviewing his life one realizes the profound influence that the higher educational system of Wales has had on the lives of the more ambitious members of the mining community, for without the opportunities given by scholarships and low fees at the University Bowen would probably have followed in the footsteps of his father and become a coal miner. It might be added that four of Bowen's family were students at the University College, Cardiff, during one session.

His brother, David Bowen, worked at the coal face as a miner for seven years then after four years at Cardiff College became the head of the Mining Department at the University of Leeds at the age of 25. David Bowen since the 1914-18 war had been at the Bar and was made K.C. in 1938. To him the writer of this obituary is indebted for details of Bowen's early life.

EZER GRIFFITHS.

NORMAN ROBERT CAMPBELL

IN the death of Norman Campbell, which occurred on 18th May 1949, at the age of 69, the Physical Society has lost one of its outstanding Fellows. It is true that, because of ill-health, he had not been seen at meetings for ten years and more, but he continued to the very last to make characteristic contributions to the *Proceedings*, and these cannot fail to have reminded Fellows of the times when a paper by Campbell would fill the lecture theatre with an

audience that found his hard-hitting controversial attack on some respectable piece of orthodoxy most stimulating. He had a stammer, which must have been a severe handicap, but he certainly didn't let it stop him; and indeed it sometimes seemed to throw into stronger relief his burning enthusiasm for that particular facet of truth that was engaging his attention at the moment.

And here let it be noted that, though he could show an almost fanatical zeal in certain directions, he was no narrow specialist, but a man who delighted in science in all its aspects. Educated at Eton, and at Trinity College, Cambridge, he worked in the Cavendish Laboratory under the leadership of J. J. Thomson for several years, becoming a Fellow of Trinity College in 1904. He contributed the chapter 1903-9 to *The History of the Cavendish Laboratory*, and in so doing gave a good account of his interests at this period; ionization, radioactivity, penetrating radiation etc. He was appointed to the Cavendish Research Fellowship at Leeds at the time when W. H. Bragg was professor there, and continued on more or less the same lines of research, but he also wrote his *Modern Electrical Theory*, which had a great success, and made his name very familiar to students of that generation. He became an Honorary Fellow of Leeds University in 1913 and seemed well set for an academic career until the first world war brought a complete break by immersing him in applied physics.

Before this occurred, however, he had already shown signs of an interest that certainly forms no part of the Cavendish tradition; an interest in what might loosely be called the logic or philosophy of physics, though Campbell would probably have repudiated both terms. It first appears in a small book on *The Principles of Electricity*, which he wrote for the series of sixpenny books published by Jack, The People's Books. This must have puzzled many a novice. Electricity was presented to him not as that by which marvels have been accomplished, but that which provides wonderful opportunities for clear hard thinking about the first principles of science. Though Campbell subsequently followed many other lines of work, this was probably the one that he had most at heart, and he never abandoned it, although its reception seldom brought him anything but discouragement. In this vein we find his great book *Physics, The Elements* (1930) and the smaller ones *Measurement and Calculation* (1928) and *What is Science?* (1921). All this work leaves the reader with no doubt that to Campbell the primary object of science was the enrichment of the human mind rather than the comfort of the human body. How strange then that he should have been content to spend the greater part of his working life in an industrial research laboratory! The old term 'natural philosopher' seemed to fit him so well that it was hard to see in him also the physicist as the handmaid of industry. But the fact is that he joined the staff of the Research Laboratory of the General Electric Company Ltd. on its foundation in 1919 and worked there until he retired in 1944. Moreover he obviously derived much satisfaction from his position as a member of this large and powerful research organization. This no doubt was partly because he had seen the organization grow up and had played an important part in establishing in it a genuinely scientific tradition, but it was probably partly also because he liked to have a hand in the ordinary workaday world. How else can one explain that after he had retired he voluntarily undertook to prepare a survey of systems of traffic lights, on finding out that one was needed.

As mentioned earlier, it was the War of 1914 that first deflected him towards applied science. He left Leeds for the National Physical Laboratory where he worked on magnetos and spark ignition with C. C. Paterson, work which after the war they published in the *Proceedings*. When Paterson left the N.P.L. in 1919 to organize research laboratories for the G.E.C. Campbell accompanied him and remained with him for the rest of his working life. The two men, so very different on a casual acquaintance, gave ample evidence that they valued the association highly. Paterson's original team indeed gave the impression of being a closely knit band of pioneers. One might be surprised to find Campbell in it, but, once in, it was not difficult to understand his reluctance to drop out, and it was only ill-health that eventually forced him to do so.

He worked on the clean-up of gases by the electric discharge, took a leading part in the development of the caesium photoelectric cell, and wrote with Miss Ritchie a book on photoelectric cells. In this work we see Campbell as an experimental worker at the bench. With his tall lean form and intense concentration on the idea of the moment he could, when hard on the trail, be somewhat formidable. A laboratory assistant fresh from school admitted that she found him rather frightening, but for all that the impression that remains

is one of great personal charm. His working bench steadily accumulated a litter of discarded glass tubing, which it was absolutely forbidden to touch. When it reached a depth at which something simply had to be done, he himself would do it. Later on illness took its toll and he had to turn more and more to work on paper as distinct from work on the bench. He was to be found working on many problems in applied science; photometry, colour matching, statistics, 'noise' in valve circuits, and even on the drafting of patent specifications. He confessed that this last, although quite interesting until he had mastered the technique, did become exceedingly boring, and probably nothing but the sense that it at least allowed him to keep his shoulder to the wheel, kept him at it.

All this time he maintained a passionate interest in the fundamentals of physics and produced from time to time the books that have already been mentioned and papers of a similar interest. He himself was convinced that his work on the roots of the subject was far more important than the endless pursuit of novelties, and that sooner or later the need for a far more critical attitude than is customary would be recognized, but he was under no illusion that physicists in general or even his own associates shared this opinion. There were undoubtedly times when he felt like a voice crying in the wilderness, but lest this should give a distorted impression of him it would be well to quote from an unpublished article he wrote in the 1930's. He tells how on a day in 1913 a copy of the *Phil. Mag.* fell out of his bookcase and lay open on the floor.

"Some algebraic formulae caught my eye It was part of a paper by a Mr. N. Bohr of whom I had never heard I sat down and began to read. In half an hour I was in a state of excitement and ecstasy, such as I have never experienced before or since in my scientific career. I had just finished a year's work revising a book on *Modern Electrical Theory*. These few pages made everything I had written entirely obsolete. That was a little annoying, no doubt; but the annoyance was nothing to the thrill of a new revelation, such as must have inspired Keats' most famous sonnet. And I had so nearly missed the joy of discovering this work for myself and rushing up to the laboratory to be the first to tell everyone else about it!

Twenty years have not damped my enthusiasm"

His friends will see the authentic Campbell in this fragment, and it is a good example of his writing at its best. One other example may be given; it is from a letter to the present writer, dated November 1944.

"In March this year I carried into effect my long laid plans for retiring from all scientific work and adopting an unintellectual life in the country Our plans for a peaceful retirement were rudely shattered. After having endured the full length of the London blitz and having so many hairbreadth's escapes that we regarded ourselves as having charmed lives, we were caught by a stray bomb, jettisoned from a plane that was being chased, in an area that had never had a bomb before. Our house and almost all its contents were completely destroyed; I have lost all my papers and nearly all my books; and have had to take refuge here with some kind friends and to resume scientific work in order to give myself occupation."

The quotation speaks for itself. It was followed by the suggestion that we should collaborate on a project that he had much at heart. The outcome can be found in the *Proceedings*, but it is Campbell's extraordinary fortitude in adversity that stands out most in these closing years. He and Mrs. Campbell faced the loss of their son on active service, the loss of their home and severe injury to Mrs. Campbell, but they could still enjoy Chaucer and he could work on the principles of electrical measurement. The death of Mrs. Campbell was a staggering blow, but once more he marshalled his forces, returned to the attack on the principles of magnetism, and looked round for a job (unpaid) in which he could help some scientific organization. It was in this way that he came to make a survey of traffic lights. Having settled down with his daughter near Nottingham, he came up to London for The Physical Society Exhibition of 1949, but the effort was too much for him and he died soon after his return, a man of science to the last.

It would have been pleasant to have had his photograph in the *Proceedings*, but it is characteristic of the man that none could be found.

L. H.

PROF. C. L. FORTESCUE

PROFESSOR C. L. FORTESCUE's death on 22nd September, only three years after his retirement from the chair of electrical engineering at the Imperial College of Science and Technology, London, came as a shock to his many friends and former colleagues and staff. He was born in 1881 and educated at Oundle School and Christ's College, Cambridge, where he obtained a First Class in the Mechanical Sciences Tripos. He played an active part in school and college life, was a keen oarsman, and reached the 'trial eights' while at Cambridge. This keen interest in student activities, and particularly rowing, never diminished, but probably increased during his later years, and he was rarely away from the river when Imperial College crews were racing.

On leaving Cambridge he spent some years at Messrs. Siemens, dynamo works and as instructor in electrotechnics and applied mechanics at H.M. Torpedo and Gunnery Schools, Portsmouth. During 1911-22 he was professor of electrical engineering at the Royal Naval College, Greenwich, and during this period, in addition to his academic and administrative duties, he carried out experimental work connected with wireless telegraphy during the first world war.

In 1922 he was appointed professor of electrical engineering at the Imperial College of Science and Technology, London. He was the third to occupy this position since 1885, the two previous occupants being Professors W. E. Ayrton and T. Mather. During the whole of the twenty-four years that he was professor and head of the Electrical Engineering Department he was indefatigable in his work for the College, and never seemed to begrudge the long hours spent in the Department. He was always accessible to the students and ready to help them in any way that was possible. He was, until his retirement in 1946, President of the College Radio Society, which involved giving twenty-four Presidential addresses, and he gave considerable time and energy to their preparation so as to present something which was interesting and instructive to the members of the Society. As a lecturer and teacher he was probably more successful with the advanced and postgraduate students than with those in the earlier years, who, in many cases, found his methods of dealing with electrical engineering somewhat difficult to appreciate. He will, generally speaking, be remembered at Imperial College for his very strenuous work in connection with the teaching of the subject of telecommunications at the undergraduate stage and the development of facilities for study and research in this subject for postgraduate students.

This development was so successful that it can be stated that the Electrical Engineering Department of the Imperial College provided a more thorough course of education in the subject of telecommunications than any other college or university in Great Britain, and students from all over the world were attracted to both the undergraduate and postgraduate courses.

As a professor of the University of London, he took very seriously the many duties which fell to his lot, and devoted a great deal of care and attention to the work of the many committees and boards of which he was a member. He served on the Senate of the University, was Dean of the Faculty of Engineering and Chairman of the Board of Studies in Electrical Engineering.

In 1942 Prof. Fortescue was elected President of the Institution of Electrical Engineers, which was a fitting recognition not only of his professional standing and academic qualifications, but also of the devoted work he had done for the Institution for very many years.

Prof. Fortescue very rarely took any rest from his work, and since his retirement in 1946 he had been active in the service of the University and also at Southampton Technical College in connection with the development of courses in telecommunications, electronics and electrical measurements. He is survived by his wife, son and daughter.

A. R.

DR. E. P. HARRISON

E. P. HARRISON, born in London in 1877, died suddenly on 6th May, 1948. He was a Fellow of the Physical Society from 1900 to the time of his death. He was also a Fellow of the Institute of Physics and a Fellow of the Royal Society of Edinburgh.

Whilst a student in University College London he won an 1851 Exhibition Research Scholarship in physics and proceeded to the University of Zurich where he obtained his Ph.D.

In 1904 he went to King's College, Cambridge, and did two years research in physics at the Cavendish Laboratory. Later, he was appointed professor of physics in the Presidency College, Calcutta, where he was also head of the Observatory.

In 1923 he returned to England as Chief Scientist in the Mine Design Department, H.M.S. *Vernon*, Portsmouth, a research post under the (then) Director of Scientific Research, Admiralty. He remained in this post until 1937 when he retired on reaching the age limit. Much of his work for the Admiralty remained unpublished, on grounds of secrecy. His main interests however, centred around the application of magnetostriction to acoustics and magnetic measurements. He was the author of numerous original papers on these subjects. The principle of his 'A-E' magnetometer, employing the change of electrical impedance of a mumetal wire in longitudinal magnetic fields, was used extensively in degaussing ranges during the war to measure the magnetic condition of ships.

On retiring from Admiralty service in 1937 he joined the research staff of Henry Hughes & Sons who at that time were concerned with the Admiralty magnetostriction echo depth sounder. Harrison's fundamental knowledge of magnetostriction helped considerably in the development of this device.

His pleasant personality and his persistence in the face of great difficulties will be remembered by his colleagues in the Admiralty.

Dr. Harrison is survived by his wife and one son. Another son, Robin, was lost in a 'Sunderland' seaplane which failed to return to its base in the early days of the war.

A. B. WOOD.

RICHARD ALBERT HULL

THE death of Dr. R. A. Hull, who was killed in a climbing accident on Mont Blanc on 22nd August 1949, robs Oxford of a distinguished experimental physicist and an outstanding successful college tutor. His friends will remember him as a man of plain-spoken sincerity and steadfast loyalty.

Richard Albert Hull was born on 27th March 1911 at Church Gresley, Derbyshire, and was educated at Ashby de la Zouch Grammar School. He entered Oxford in 1929 as an Exhibitioner of St. John's College, and read mathematics and physics, taking a first class in physics in the Final Examination in 1932. His first research was an experimental study of the photoelectric properties of transparent thin films of caesium and rubidium. The work was an exacting test of Hull's experimental skill, which was recognized when he was elected a Senior Scholar of Christ Church in 1934. He did not however find it an altogether satisfying field of work, and by the time he submitted his thesis for the degree of D.Phil. he had already turned to low temperature research. Hull took part in the investigation of the range of temperatures below 1°K . accessible by the new method of magnetic cooling. This was a field exactly suited to his temperament and gifts. Interested in experimental technique for its own sake, he enjoyed the exploration of the problems of adiabatic demagnetization, of temperature measurement, and of thermal insulation, which demanded solution before successful experiments could be made in the new field.

The conditions of work were often harassing and difficult. Experiments usually lasted from fifteen to twenty hours, and any accident or delay meant working throughout the night. Specific heat measurements in particular involved long hours of monotonous galvanometer reading. Hull was fortunate in a power of concentration which enabled him to keep command of an experiment in the most trying conditions. He worked himself hard, and the technique of work below 1°K . established in Oxford in 1939 owed much to his patient and ingenious experiments. He had published papers on the vapour pressure of liquid helium, on the attainment of low temperatures by evaporation of helium, and on magnetic effects in iron ammonium alum, one of the most important substances for the magnetic method. In addition, work had been completed on the properties of manganous ammonium sulphate, from which the absolute scale of temperature below 1°K . could be established by using this salt as a magnetic thermometer.

The war interrupted all this. Hull joined the research group formed at Oxford by the Admiralty to work on problems of radio waves of very short wavelengths, and began a study of millimetre waves. His outstanding contribution was the design and construction in 1941 of a velocity modulation oscillator for 9 mm. wavelength, which was used for investigations of propagation and reception of millimetre waves. This was probably the first continuous wave generator in the millimetre wave region.

Hull had been appointed a University Demonstrator and Lecturer in Physics in 1939, and as the war's demands increased he became more and more taken up with the teaching and training of physicists. He took a large part in the working at Oxford of the State Bursary scheme, and the Hankey radio training scheme. In 1944 he spent a term as a visiting professor at Harvard, assisting in the U.S. Navy's radio training scheme there. In the same year he was elected a Fellow of Brasenose College. When undergraduates poured back to Oxford at the end of the war, Hull found himself the only science tutor in a large college, at least a quarter of whose undergraduates were reading science subjects. For a time he undertook the general care of all scientific studies in Brasenose, at the same time playing his part in the re-organization of the University physics department, where the number of students was treble the pre-war entry. In 1948 he became a member of the General Board of the Faculties, and soon became an accepted and vigorous representative of the science faculties. These activities made heavy inroads on the time he could give to research, but he returned to the field of work below 1°K . in which he had been engaged before the war, and at the time of his death was making experiments on the specific heat of liquid helium in this temperature range.

He was a rock of reliability in any matter entrusted to him. Any problem he took up, whether it was the troubles of a new pupil or the entrance requirements of the University, was considered long and deeply. When his mind was made up, he would speak it, even though it meant saying the difficult and unpopular thing. Something of this same deliberate quality was the secret of his success as a scientist and a teacher: never to accept a partial explanation and, when a satisfying answer was found, to reduce it to its clearest terms. He had many friends from a wide range of University life, from the hockey field, the Bach choir, and the Music Club, as well as those who enjoyed his company and quietly ironical humour in the closer society of common room. His nature however was perhaps too reserved for many to know him intimately. Accustomed to rely upon himself, he would seldom reveal his personal feelings or problems. Yet many brought their problems to him, and found him generous of time and trouble to help them.

He was an enthusiastic and very experienced mountaineer. Climbing was his passion; there can have been few vacations in the last ten years when he did not escape, at least for a short time, to the mountains. At the end of a hard term's work he would slip away to Scotland or Wales, or to the Alps, to return bronzed and refreshed, often with beautiful photographs of the rocks and mountains to delight his friends. He would persuade as many as he could to join him in the sport he loved, and was one of the leaders of this year's Alpine Club party to Switzerland. The climb on which he lost his life was to have been the last of the season—an ascent of Mont Blanc by the Brouillard ridge.

In 1937 Hull married Miss Judith Moore, like himself a research worker in the Clarendon Laboratory. She died in 1943. They leave one daughter.

A. H. C.

C. N. H. LOCK

CHRISTOPHER NOEL HUNTER LOCK was born on 21st December 1894. He was a Scholar at Charterhouse School and in 1912 was awarded a Major Scholarship at Gonville and Caius College, Cambridge. He became a Wrangler in Part II of the Mathematical Tripos, with distinction in Schedule B, in 1917. He was awarded a Smith's Prize in 1919 for an essay on External Ballistics and in the following year was elected a Fellow of Gonville and Caius College.

During the 1914–18 war and for two years afterwards, Lock worked on problems of external ballistics at Whale Island, Portsmouth, and at the National Physical Laboratory, Teddington. The results of an elaborate series of experiments on projectiles undertaken, in collaboration with R. H. Fowler, E. G. Gallop and H. W. Richmond, during the course of this work are given in two classical papers on "The Aerodynamics of a Spinning Shell", published in *Phil. Trans. A*, Vols. 221 and 222. In 1921 he wrote, with R. H. Fowler, a paper on "The Origin of the Disturbances in the Initial Motion of a Shell" (*Proc. Camb. Phil. Soc.*, Vol. 20) and in 1922, also with R. H. Fowler, a paper on "Approximate Solutions of Linear Differential Equations" (*Proc. Lond. Math. Soc.*, Vol. 20).

In 1920 Lock was appointed to the staff of the Aerodynamics Division at the National Physical Laboratory. For about two years he worked on general aerodynamic problems such as the cushioning effect of the ground on aircraft performance and the lateral control

of aircraft, and then began his well-known work on airscrew problems, a subject on which he became an outstanding authority. He carried out both theoretical and experimental researches which contributed much to the development of modern airscrew theory and design. His researches, most of them described in reports published by the Aeronautical Research Committee, cover a wide field and include experiments with a family of airscrews, interference between airscrews and bodies, experimental verification of the independence of the elements of an airscrew blade, a consideration of the accuracy of the vortex theory of an airscrew, an application of Goldstein's theory to airscrew design, the flow around an airscrew in the vortex-ring state, and a graphical method for the calculation of airscrew performance. He also made wind-tunnel experiments on a model autogyro and contributed to the development of autogyro theory.

From 1939 until his death Lock organized and supervised the work undertaken in the High Speed Laboratory of the Aerodynamics Division. He played a prominent part in the design of the high-speed wind tunnels and in the development of the apparatus needed for experimental research, and his wide knowledge of the theory of compressible fluid flow was put to good use in the solution of difficult research problems. He took a very active part in all the work undertaken in the High Speed Laboratory and gave unstinted help to all his colleagues. His individual investigations include development of the pitot-traverse method for the determination of profile drag at high speeds, estimation of tunnel-wall interference at compressibility speeds, calculation of the ideal drag due to a shock wave, and a consideration of the effect of sweep-back on the performance of an aerofoil at high Mach numbers.

Lock was a Fellow of the Royal Aeronautical Society and a Fellow of the Physical Society. He was a member of the Power Plant Committee, the Helicopter Committee the Fluid Motion Sub-Committee and the Engine Aerodynamics Sub-Committee of the Aeronautical Research Council. He gave two lectures to the Royal Aeronautical Society, the first on "Airscrew-Body Interference" in 1929, and the second on "Problems of High-Speed Flight as affected by Compressibility" in 1937.

Lock died on 27th March, 1949, after a short illness. He suffered for many years from asthma and the after-effects of infantile paralysis, but he bore these bodily disabilities with great fortitude.

He married Lilian Mary Gillman in 1924 and his two sons, Robert Christopher and John Michael, are research students at Gonville and Caius College.

A. PAGE.

WILLIAM BLAIR MORTON

PROFESSOR W. B. MORTON, who died in his native city of Belfast on August 12th at the age of 81, was one of the many physicists of his generation whose formal training had been principally in mathematics. He attended Queen's College, Belfast—then part of the Royal University of Ireland—from 1886 to 1889, reading mathematics under John Purser, and natural philosophy under J. D. Everett, and St. John's College, Cambridge, from 1889 to 1892, when he was eighth wrangler. He was profoundly impressed by Purser's methods of teaching, and strove in later years with conspicuous success to make his own lectures clear and interesting. Morton followed Everett in the chair of natural philosophy at Belfast in 1897 after being his assistant from 1892. He retained the chair until he was retired in 1933 on reaching the age limit. On appointment, one of his first acts was to increase the number of lectures given to undergraduates. Shortly afterwards, with the help of J. Wylie, he extended the laboratory classes. A weekly physical colloquium was started in 1904, after a visit to Germany. The conversion of Queen's College in 1909 into the independent Queen's University of Belfast brought no immediate change beyond the alteration of his title to Professor of Physics, but funds became available in 1911 for the erection of a new physics laboratory. This was designed by Wylie, and completed in 1914. Morton had to meet considerable criticism of its size, which was then considered excessive by some colleagues, and for various periods part of the basement was surrendered to the University's works and engineering departments, and part of the tower to chemistry. Good provision was made in the building for experimental research, although Morton himself took little part in it, preferring to devote such time as he had free from lectures to original theoretical work, and the history of mathematics and physics. Similarly, whilst he was very interested in the undergraduate laboratories, the detailed running of these was

left to Wylie. Although he was author or joint author of a considerable number of papers, he refrained from publishing much which might well have gone into print, preferring to talk about it in colloquium or postgraduate lectures. The present writer and others who were in Belfast in the late 1920's have much reason to be grateful to him for his expositions of quantum theory, which he recognized as being largely a transcription into a new context of the classical work of Rayleigh with which he was intimately acquainted.

Morton was a Fellow of the Royal University of Ireland from 1897 until its dissolution in 1909, a Member and for a time Vice-President of the Royal Irish Academy, and a Founder Fellow of the Institute of Physics. After his retirement, Queen's University conferred the degree of Doctor of Science on him, and he was made a member of Senate, its supreme governing body. He did not, however, seek for honours or public recognition, and was probably most happy in his work when he was in his department with his staff and students. To those who had the privilege of collaborating with him in any capacity his death has been a personal loss.

K. G. E.

ALAN FARADAY CAMPBELL POLLARD

THE death, on 15th August 1948, of Professor Alan Faraday Campbell Pollard deprived the world of science of one of the few who behold with increasing disquiet the spectacle of a community which spends money lavishly on research and the building up of knowledge, but gives little thought as to the ways in which this knowledge can be made accessible. Some of his favourite examples included the neglect, for 35 years, of the work of Mendel; the patent (1876) of Penaud and Gauchot which described many of the features of the modern aeroplane 27 years before the work of the Wright brothers; and the 81 years between the discovery of 'Herapathite' and the perfection of Polaroid by E. H. Land. Pollard was an enthusiastic advocate of the Universal Decimal Classification and was the first President of the British Society for International Bibliography. For many years he kept up to date a U.D.C. catalogue of the *Proceedings of the Physical Society*. He served on the Council of the Physical Society from 1933-37.

Born in 1878, the son of Lieut.-Colonel B. H. Pollard of the Indian Staff Corps, he studied engineering at University College, London, and was for some time assistant to Professor Karl Pearson, who had a great influence in strengthening his passion for clarity and precision, and who gave him a bent towards the scientific study of instruments. After a period with Nobels Explosives Ltd. and war service (1914-18) with the Royal Scots Fusiliers, the R.A.F., and the Ministry of Munitions, he was appointed Professor of Instrument Design in the Technical Optics Department at the Imperial College and held the last-named post until his retirement from the College in 1943. He continued, however, to work as consultant and scientific adviser to various industrial undertakings until the time of his death.

He was very critical of the usual industrial approach to the design of scientific instruments and based his own ideas on the kinematical theories of Clerk Maxwell, which (used in original ways) led him to ingenious designs of various instruments such as interchangeable nosepieces for microscope objectives and other things. He inspired many students with something of his own enthusiasm, and explained some of his ideas in a small book, *The Kinematical Design of Couplings in Instrument Mechanism*, published by Messrs. Adam Hilger Ltd.

Many will regret that the Chair of Instrument Design has not been filled since his death. The design of instruments as a specific field of study has seemed to some inadmissible, the brewer preferring to design his colorimeter and the surveyor his theodolite. Yet if really adequate instruments are to be made there are fundamental principles of mechanical and optical design the neglect of which makes for failure instead of success. At a time when the scientific instrument industry of this country is making efforts to hold its own against strenuous foreign competition, our instrument industry may yet regret that Pollard's missionary efforts in respect of instrument design have not been followed up.

He was a friendly and helpful colleague; though he deplored the unsystematic and happy-go-lucky methods of some of us, he was always ready to listen and do his best to help. He married Gabrielle, daughter of Mr. Frederick Urwick, in 1915, and had a son and a daughter.

L. C. MARTIN.

WILLIAM GEORGE PYE

WILLIAM G. PYE, born on 27th October 1869, at Battersea, died at Bexhill on 13th October 1949. He was a son of William Thomas Pye, an expert maker of mechanical models. About 1878, A. G. Dew-Smith began making physiological instruments in Cambridge. In January 1881 Horace Darwin joined him as partner, to form the Cambridge Scientific Instrument Co. In December 1880 they offered W. T. Pye the post of foreman in a workshop they were to start on 1st January 1881. After a trial he was, in 1882, fully established as foreman. In 1895 the Directors of the then newly incorporated company made him Secretary: he had been Manager for some years. The company moved to St. Tibb's Row in 1881. When W. T. Pye came to Cambridge, from Battersea, his son attended the Higher Grade School. Then for a short time he taught at the Chesterton Church School and later did instrument work under his father. When W. T. Pye left, about December 1898, Mr. R. S. Whipple succeeded him as Secretary.

About 1886, W. G. Pye, now a trained youth, went to Elliott Bros., London. In 1892 the post of mechanic at the Cavendish Laboratory became vacant. At the opening of W. G. Pye's works at Chesterton (see below), Sir J. J. Thomson said:—

"It was Mr. Dew-Smith who spoke to him about Mr. W. G. Pye, when they were seeking a new head of the workshop; he had every reason to be grateful for his advice to get Mr. Pye, if he could. He did get him, and they in the Cavendish Laboratory owed a great deal to him. He immensely improved the workshops and made a great many exceedingly effective instruments, and not the least he did was to train his successor, Mr. F. J. Lincoln. Mr. Pye, in addition to his great skill as a workman, possessed the qualities of great energy and business capacity, and so he encouraged him to start as an instrument maker."

I began to teach at the 'Cav. Lab.' in 1888. As a boy I did a lot of mechanical work. I naturally appreciated Pye's 'new broom' and cordially echo the words of 'J. J.' There was leeway to make up. Maxwell made no provision for a workshop; he discovered his mistake and made a small beginning. When Pye came, in September 1891, the equipment was still meagre.

By 1892 the number of students had grown and class apparatus was needed. Pye made many things for the Medical Students—all very simple. Great additions for my own Class and for the Advanced Class were made by him or, under his direction, by Lincoln, who started as a 'lab. boy' in 1891. With his first-rate mechanical knowledge, instinct for good design, handiness and skill of hand, Pye was very helpful. Research was developing and he made much of what the researchers could not 'pinch' from the Practical Classes or make of 'sealing wax and string'.

Many research students, on leaving the Laboratory, wanted apparatus like that used there. To meet this demand, W. G. Pye, about 1895, set up a small workshop in St. Andrew's Street, with his brother, Frederick W. Pye, as foreman. On leaving the 'Cav. Lab.', in 1902, he equipped a workshop in a granary of a disused water-mill. He prospered and, in 1913, built works at Chesterton. In 1921 these works, which had grown, were attacked by the 'Radio Beetle' and a rapid expansion followed. In 1927, the works and the radio business were bought by a company, under the name of Pye Radio Ltd., now Pye Ltd.

Pye and his younger son, H. J. Pye, then built the "Granta" works in Newmarket Road. Here the manufacture of electrical and mechanical apparatus for Practical Classes was continued by W. G. Pye & Co. Ltd. Mr. W. G. Pye soon became a 'sleeping partner'; the business was then run by H. J. Pye. In 1930 W. G. Pye left Cambridge and later went to Bexhill. There he took a leading part in the "Senlac Metal Casements Co." till early in 1949. In 1946 W. G. Pye & Co. Ltd. was amalgamated with Pye Ltd.*

Many things devised for my Class were put on the market, often in improved forms, by Pye, and had a wide distribution and appeared in textbooks. The Conductivity of Copper apparatus stimulated a girl to write: "Conductivity for Heat, discovered by Dr. Searle". When we were settling a design, a flashing suggestion from Pye sometimes altered the whole plan. I knew the destructive ways of students and demonstrators, and Pye loathed things which 'came ungummed'. Sturdy and reliable apparatus was the result.

It is on record that I was pleased when, in 1911, I saw six sets of apparatus for the Mechanical Equivalent of Heat at Harvard University.

* Up to this point some of the dates given above may be only approximate.

From 1892 I had the friendship of W. G. Pye, and later of his son. We used to seek each other's help. They showed me much kindness in my work in Physics.

Sir J. E. Townsend, F.R.S., who began research in 1895, testifies :—

"I am sure that the Research Students who worked in the Cavendish when Pye was head instrument maker feel that they are much indebted to him. I certainly appreciated very much the assistance he was to me. The cheerful way he undertook to make apparatus of various designs contributed substantially to the success of the research work."

C. T. R. Wilson, F.R.S., writes :—

"I just remember him as being always most kind and helpful in his difficult task of supplying one's requirements from the Laboratory workshop."

In 1910 Pye went for his health to Torquay. With my wife and me he paid two or three visits to Oliver Heaviside.

I could not be at the opening of the Chesterton works in 1913. I wrote to Pye :—

"You have the happy power of making a feeling of sunshine around you. This will go far to secure in the future, as it has done in the past, the loyal and enthusiastic co-operation of all those who help you in any way."

In July 1892 he married Miss Annie E. Atkins, who survives him. The surviving children are : (1) Donald Walter, of New College, Oxford, now Senior Master at Llandoverly College, (2) Marjorie Irene, and (3) Harold John, of St. John's College, Cambridge, who attended my Practical Class.

G. F. C. SEARLE.

MAX PLANCK *

UNDOUBTEDLY Planck, who died at Göttingen on 4th October 1947, was one of the greatest of all those who ever turned their minds to scientific inquiry. His achievements place him among the three or four most outstanding men of science of this or any other time and, quite apart from his greatness as a contributor to our knowledge and comprehension of physical phenomena, he was a man of perfect integrity and great nobility of character, the like of whom rarely appears on our horizon.

Max Karl Ernst Ludwig Planck was born on the 23rd April 1858 at Kiel, where his father was a professor in the faculty of law. He was taken in his early years to Munich and was a pupil at the Maximilian Gymnasium for four years. In his recently published *Wissenschaftliche Selbstbiographie* † he remembers with affectionate appreciation a mathematical teacher there : one Hermann Müller, who was evidently a strong contributory influence in inclining him to take up the study of physics.

With the *abitur* of the Gymnasium Planck entered the University of Munich at the age of seventeen, where he studied under von Jolly, Ludwig Seidel and Gustav Bauer. After the manner of the German student, who was wont to migrate from one university to another, Planck spent a year at Berlin attending the lectures of Hermann von Helmholtz and Gustav Kirchhoff. He was promoted Doctor of Philosophy (*summa cum laude*) at Munich in 1879, his dissertation being entitled *Ueber den zweiten Hauptsatz der mechanischen Wärmetheorie*. Thereafter he became *Dozent*, the subject of his *Habilitationschrift* being *Gleichgewichtszustände isotroper Körper in verschiedenen Temperaturen*. In 1885 he was called to the university of his native town, Kiel, as Professor *Extraordinarius*, having in the meantime completed the now well-known essay on the Principle of Conservation of Energy, which he submitted for the prize offered by the Göttingen Philosophical Faculty. Planck was adjudged second and one has a suspicion that the Faculty of that day were perhaps hardly capable of appreciating his essay at its full value.

In the spring of 1889 he was called to Berlin (as *Extraordinarius*) to succeed Kirchhoff, on the recommendation, or initiative, of Helmholtz. He became Professor *Ordinarius* in 1892 and filled the office of Rector during the Semester of 1913–1914. His inaugural address was entitled *Neue Bahnen der physikalischen Erkenntnis*. In 1926 he retired and was succeeded by Erwin Schroedinger. He was awarded the Nobel Prize in 1918, taking as the subject of his address to the Royal Swedish Academy (on the 2nd June 1920) The

* This was received soon after the December 1948 issue of the *Proceedings* had gone to press.

† I am indebted to Professor Flint for providing me with a copy of this.

Origin and Development of the Quantum Theory. The Royal Society elected him a Foreign Member in 1926 and awarded him its Copley Medal in 1929. In 1930 he became President of the Kaiser Wilhelm Society. He also served for many years as one of the secretaries of the Prussian Academy, to which he was elected soon after the death of Hertz in 1894.

Planck's most important work was his investigation of the nature of black body (cavity) radiation, which led to the solution of the problem of the distribution of energy in the so-called normal spectrum and, as is well known, to the discovery of the quantum of action, h , and the quantum theory. Before his attention was directed to the radiation problem, he spent many years in the study of thermodynamics and in securing its foundations. No doubt this long thermodynamical training enabled him to appreciate better than anybody else, the nature of the formidable problem which he eventually solved.

His early work on thermodynamics evoked little response. Apparently it was not even read by Helmholtz, Kirchhoff or Clausius. This is perhaps not so surprising, when we reflect that thermodynamics must have impressed them as a finished product in which only trivial modifications could be contemplated. Indeed it still retains substantially the form which Clausius gave to it. It is perhaps easier to forgive this great triumvirate than later specialists in thermal science for their neglect of Planck. His attitude to thermodynamical principles was different from that of all his predecessors. He was not hypnotized by the literal sense of the terms 'reversible' and 'irreversible'. He did not begin with reversibility at all, but by defining an irreversible process, which for him was a process which could not in any way whatsoever be completely undone (*rückgängig gemacht*). Any other processes were defined to be reversible. He seems to have been led to this attitude by studying the axiom of Clausius that heat cannot of itself pass from a colder to a warmer body. Notwithstanding Planck's criticisms this axiom is still used, quite uncritically, in quite recent textbooks.

Beyond doubt Planck succeeded in giving the second law a sound foundation, and his work, as set forth in his great book *Thermodynamik*, is rigorous; but there are alternatives to his approach, or at least one. One finds some indication of this in an unknown work by Carl Neumann (1875); but Neumann and Planck seem to have been unable to understand one another. They seem to me to be both equally sound.

In approaching the problem of the distribution of energy in the normal spectrum Planck made use of the fact, pointed out by Kirchhoff, that the character of the radiation within a vacuous enclosure, in temperature equilibrium, is independent of the nature of the emitting and absorbing material constituting the enclosing wall. He realized that this fact entitled him—in investigating the character of the radiation under temperature equilibrium—to assume any sort of emitting or absorbing systems he liked, and so he assumed what seemed to him the simplest thing, namely the Hertzian oscillator. He had for his guidance the two formulae obtained by W. Wien (1896); in particular the exponential one which Lummer and Pringsheim had shown to fit the observations in the region of very short wavelengths. Probably at that time Rayleigh had not yet developed his formula; but its experimental equivalent had emerged from the work of Rubens and Kurlbaum in the region of very long waves—the residual radiation from rocksalt, fluorspar etc. Planck's preoccupation with thermodynamical studies, as he himself has indicated, made it natural that he should study the relation between the entropy of an oscillator and its energy. Wien's formula gives for U , the mean energy of an oscillator in temperature equilibrium,

$$U = ae^{-b/T},$$

a and b being constants. Now remembering that $dU/dS = T$, it easily follows that

$$1 \left/ \frac{d^2S}{dU^2} \right. = -bU,$$

S and T meaning entropy and temperature respectively. On the other hand the results of Rubens and Kurlbaum suggested $U = cT$, where c is a constant. This in effect is Rayleigh's formula. Consequently

$$1 \left/ \frac{d^2S}{dU^2} \right. = -\frac{U^2}{c}.$$

Naturally Planck tried the experiment of combining these two in the following simple way :

$$1 \left/ \frac{d^2 S}{dU^2} \right. = -bU - \frac{U^2}{c},$$

the solution of which yields

$$U = bc(e^{b/U} - 1),$$

which is virtually Planck's final formula; but the problem of deducing it from fundamental principles still remained. In order to make progress in this direction Planck turned to Boltzmann's H theorem and its implication of a functional relationship between the entropy of an assemblage and the probability of its state. He was thus led to equate the entropy of an assemblage to the product of a universal constant and the logarithm of what he called the thermodynamical probability. This universal constant, which he represented by k , is usually called Boltzmann's constant; but was in fact first introduced by Planck. Indeed, as Planck has said, his theory is distinguished from that of Boltzmann precisely by the assumption or premiss that a definite finite number k , actually exists, with the property that its product with the logarithm of the thermodynamical probability is equal to the entropy.

The rest is very simply told. In any case, whether a departure from classical principles were contemplated or not—and probably Planck did not have this in mind at first—it was necessary to divide the extension in phase of the oscillators—each assumed to have one degree of freedom—into elements with the dimensions *energy* $\times$ *time*. The classical procedure necessarily involves that these elements approach the limit zero. Planck realized however that this gave the oscillators the average energy $U = cT$, which could only fit the observations in the region of very long waves. Thus he was forced to endow each element of the phase space with a precise numerical value, *different from zero*. He represented this element by the symbol, h , and called it the *elementary quantum of action*. It was a consequence of this that the energy of each oscillator took the form

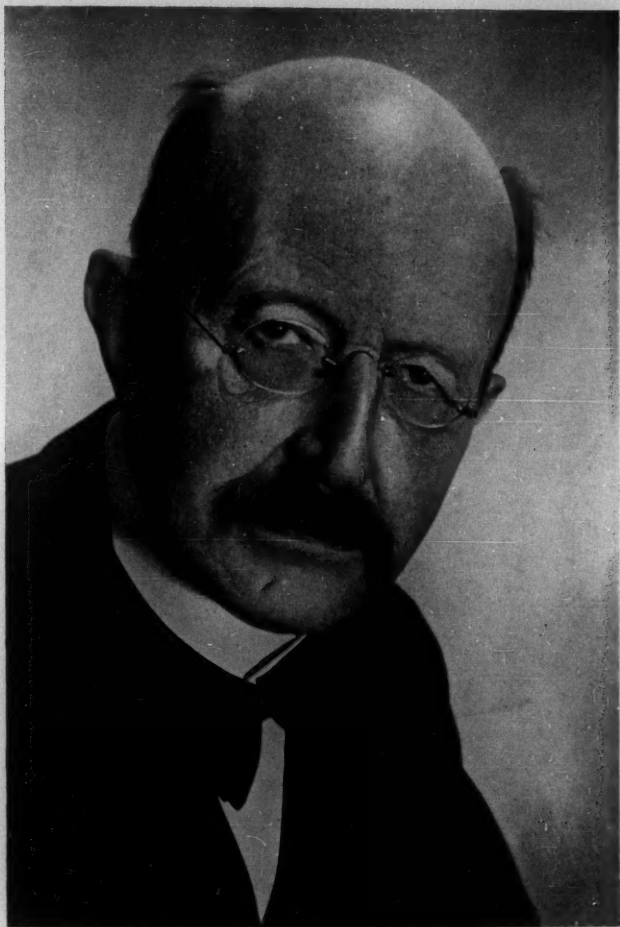
$$\text{Integer} \times h\nu,$$

or, in his later and slightly modified theory—in which he assumed that *energy emission only* was discontinuous—

$$(\text{Integer} - \frac{1}{2}) \times h\nu.$$

Just as the constant k was suggested to Planck by Boltzmann's theory, so the much earlier work of Sir William Hamilton might have suggested the constant h . It appears later, in wave mechanics, not only as the quotient of energy by frequency, but also as the quotient of momentum by wave number. Indeed the study of Hamilton's analogy between geometrical optics and classical mechanics might well have suggested a more general mechanics analogous to optics or wave propagation in the widest sense.

Planck communicated his theory to the German Physical Society and it was published in their *Proceedings* and in the *Annalen der Physik* at the very close of the century. Anyone who can remember the state of physical science in 1900 and the outlook of physicists at that time will appreciate that Planck's theory of radiation was at first regarded as very eccentric if not quite mad; but when one re-reads it one realizes that it is an almost inevitable consequence of the actual facts of black-body radiation already known. A very strong impression was made by Planck's computation of Loschmidt's (or Avogadro's) number and consequently of the elementary ionic charge, which he found to be 4.69×10^{-10} E.S.U. Rutherford and Geiger found 4.65×10^{-10} E.S.U. a year or two later by counting alpha particles and estimating their total charge. Planck's result is remarkably close to 4.803×10^{-10} E.S.U., which I think is the present estimate. He computed h to be 6.55×10^{-27} ergs $\times$ sec., a value which has not been appreciably modified since, and k he found to be 1.34×10^{-16} ergs per $^\circ\text{K}$., now estimated to be 1.372×10^{-16} ergs per $^\circ\text{K}$. Many important consequences soon followed from Planck's theory of *Quanten-emission* even before a coherent quantum mechanical theory was built up: Einstein's photoelectric theory; the dependence of specific heats on temperature, especially Debye's brilliant treatment of the atomic heats of solid elements; Bohr's theory of the hydrogen atom and of the extranuclear structure of atoms and the theory of spectra; the contributions of Sommerfeld, Epstein, Kramers and others and then the triumph of quantum mechanics (L. de Broglie, E. Schrodinger, W. Heisenberg, M. Born and P. Dirac).



Dr. Max Planck.



WILLIAM NELSON STOCKER

Planck was twice married and is survived by his second wife (Margaret, *née* von Hoesslin) and a son. Until the outbreak of the first world war he might indeed have been described as a happy and fortunate man, particularly fortunate in his teachers, in his academic career and in making one of the greatest of all scientific discoveries. But then calamities came. His eldest son, Karl, fell before Verdun in 1916 and he lost his twin daughters. In the last war his house in the Grunewald quarter of Berlin was destroyed in an air raid and his fine library lost. He happened to be in Cassel when that town was destroyed and was buried for hours in an air-raid shelter. After the attempt on Hitler's life, on 20th July 1944, his son Erwin was arrested, condemned and executed on 23rd Jan. 1945, solely on the ground, apparently, that some of those involved belonged to his circle of acquaintances.

Planck was found in May 1945 by the Americans at Rogätz on the Elbe and conveyed by them to Göttingen where he remained till his death and where his remains now lie.

Many Fellows of the Physical Society will remember his Guthrie lecture in 1932 and his profoundly interesting and illuminating discussion of causality. He had been elected an Honorary Fellow of the Physical Society in 1924. In 1936 he attended, as a guest, the centenary celebrations of the University of London, and it was on this occasion, I believe, that the University conferred on him the degree of Doctor of Science *honoris causa*. He obviously much enjoyed his visits to St. Paul's Cathedral and to the Guildhall and the dinner in his honour at the Waldorf Hotel.

His last visit to England was in 1946, little more than a year before his death, to attend the Royal Society's Newton celebrations. He was no longer the erect and vigorous man that many of us remember; but he retained his mental clarity and alertness.

Planck was an enthusiastic musician and a fine pianist. He was also a great mountaineer and continued to make strenuous ascents of 3000 metres or more after reaching 70 years of age. I have it on the authority of Professor Sommerfeld that Planck made personal representations to Hitler on behalf of persecuted Jewish scientists. He was of course 'schroff zurückgewiesen'.

I remember him as a modest, kindly man, who, in the words of one of the greatest of his countrymen,

weit entfernt von allem Schein,
Nur in der Wesen Tiefe trachtet.

WM. WILSON.

WILLIAM NELSON STOCKER

THE name of W. N. Stocker, who died on 2nd August 1949 at Oxford, in his 99th year, will probably sound unfamiliar to most readers of the *Proceedings of the Physical Society*. Even in Oxford, where he taught physics on and off for over forty years, his name is better known to those interested in original 'Oxford characters' than to the physicists. And yet Stocker, who was born in the year Kelvin formulated the Second Law of Thermodynamics, who took Finals in the year Maxwell published his *Treatise*, and who, in his own words 'lost the habit of work' in the year Einstein announced the Special Theory of Relativity, was for the last ten years or so the *doyen* of British physicists.

William Nelson Stocker was born on 29th January 1851 at Horsforth, near Leeds, the son of the Rev. William Henry Browell Stocker. After education at Stony Stratford School he matriculated at Oxford in 1869 with the intention of reading classics and taking Holy Orders. His interest in science soon caused him to change his plans, and in 1870 he started studying mathematics and physics, gaining a First in Mathematical Moderations in 1872 and two Firsts in the Final Honour Schools of Mathematics (1873) and Natural Science (1874). It was presumably for lack of sufficient money that he had started as a non-collegiate student and remained one until 1873, when he gained an Exhibition at Christ Church, and his first five years in Oxford must have been a period of hard work, modest living and recreations chosen with an eye on a tight budget. Little is known about these years, but from the detailed entries of his account book covering the years 1869-1878 one pictures young Stocker as an undergraduate who, within his modest means, enjoyed the amenities that Oxford could offer. Boating and skating alternated with long walks in Oxford and its neighbourhood, books and music were bought for study and for leisure; then there were public concerts (the Hallé Society's concerts seem to have been regular features in those

days) and the private enjoyment of playing the piano and the organ. We also learn from this account book that the cost of his fifteen terms at Oxford, including University expenses, board and lodging, books, etc. was just under £400, a very modest sum even for those days.

He took his degree in 1874, and in the same year accepted a demonstratorship at the then new Clarendon Laboratory under Professor R. B. Clifton in preference to becoming a science master in Cheltenham. In 1877 he was elected to a Fellowship at Brasenose for life, and his death marks the disappearance of the last of these Oxford life-Fellows. In the same year he became a Fellow of the Physical Society and, having held this Fellowship for 72 years, established a record which will be very hard indeed to break. His generous contribution to the Physical Society's Holweck Fund, one of his many donations to deserving causes, will be gratefully remembered. He was one of the first members of the Oxford University Mathematical and Physical Society, founded in 1888, and although there is no record of his ever reading a paper to that Society he took a lively interest in its functions, and he was genuinely disappointed when, in November 1948, failing health prevented him from attending the dinner commemorating the Society's 300th Meeting. Stocker was a conscientious if not inspiring teacher. He was more interested in experiment than in theory, and many generations of undergraduates who passed through his hands in the Clarendon Laboratory learned from him how to carry out physical experiments carefully, honestly and accurately.

In 1883, at the age of 32, Stocker was appointed Professor of Physics at the Royal Indian Civil Engineering College at Cooper's Hill, near Egham, Surrey. He held this post for the next 18 years till 1901, and delivered regular courses of lectures on heat, electricity and magnetism, light, etc. He used to visit Oxford for occasional week-ends during these years, keeping up his old connections with Brasenose, so that when in 1901, at the age of 50, he retired from Cooper's Hill and returned to Brasenose it was like coming home. He continued to do some teaching for the College and demonstrating in the Clarendon Laboratory, but these activities were on a diminishing scale, and came to an end soon after the 1914-18 war. One important event during this period was his "around the world trip", as he called it. He went to Australia via the Indian Ocean in the summer of 1914 to attend the British Association Meeting there and returned via Canada, arriving back in England in October.

Little can be said about the last 30 years of Stocker's life, especially about his scientific pursuits. He was never much interested in new developments in physics, and with 'Einstein and all that' he did not even attempt to cope. He did, however, like to hear gossip about what was going on in the laboratories and about the doings of some of the old guard whom he still knew. A few years before his death he was shown round the new Clarendon Laboratory by the late Dr. R. A. Hull, his third successor as Physics Tutor at Brasenose. He seemed to have been impressed by what he saw in the new laboratory, but perhaps even more by the sight of the old Clarendon being partly pulled down and rebuilt. It must have given Stocker some satisfaction to see that he survived not only all his human contemporaries but even the Laboratory which he knew when it was first built.

If it is difficult to give an account of Stocker's life—partly because it was so uneventful—it is even more difficult to describe his personality without the fear of doing him an injustice. What we know about him is bound to be superficial; he had very few intimate friends—most of whom have been dead for a long time—and even to them he was never communicative. This does not mean he did not like company; in fact among his greatest pleasures were the long, friendly evenings in Brasenose Senior Common Room, and he had little patience with some of the younger generation who would 'use the place as a restaurant' and go off after dinner to meetings or lectures or—*horribile dictu*—to the laboratory to continue experiments.

To those who met Stocker in the last one or two decades the mere fact of talking to a man really belonging to quite a different age was a thrilling experience. Stocker himself was rather proud of his old age and of his reminiscences going back well over 90 years. He could remember Tsar Nicholas I being burnt in effigy at Horsforth during the Crimean War; one of his 'earlier' recollections was of Queen Victoria and the Prince Consort driving in state to the opening of the new Guildhall of Leeds, and he would delight in mentioning that his uncle, Dr. C. W. Stocker, Fellow of St. John's, had examined Gladstone in 1831. It is no wonder that after hearing such reminiscences one almost believed Stocker's classic

remark about lectures on electricity: "In my days you took a piece of sealing wax, rubbed it against your sleeve, picked up a piece of paper with it, and there was your course of electricity".

By no means a brilliant conversationalist, Stocker had the gift of a dry, biting humour which enabled him to drive home a point with a minimum outlay of words and, very often, the maximum embarrassment to the unfortunate victim. Among the many stories told about him the following quite authentic one may be given as an example. It so happened that on VJ Day in 1945 a party of only four was dining in Common Room; Stocker, two people he very much disliked, and a fourth person who, the last to arrive, greeted Stocker with the remark, "A small but select gathering for this memorable day!" After a brief pause came Stocker's grunting reply, "Yes . . . humph . . . small." In spite of this mordant wit he had an air of measured old-world courtesy, and many a young guest in the Common Room must have felt surprised and flattered at being addressed as "Sir" by an old gentleman more than 70 years his senior.

Stocker's great hobby—it was an integral part of his life—was walking. His regular daily walks at Oxford and Cooper's Hill were followed in the vacations by long walking tours during which he explored the countryside and observed the flora and geology of all parts of Britain. He always carried a pedometer and a barometer on him, and kept an exact record of the distances covered and the heights ascended. His endurance was fabulous. Even when he was over 70 he could average twenty miles a day over several weeks, and one can quite believe that in his life-time he all but covered the distance to the moon and did over a hundred ascents of Everest. These and similar data are contained in some seventy-five volumes of diaries which also present a faithful record of the weather from 1873–1949 with thermometric and barometric readings.

One might wonder why it was that Stocker, undoubtedly an able man with a capacity and a constitution for hard work and with a deep interest in nature, never made a name for himself in the world of physics. One should remember, however, that in 1874, when Stocker became a demonstrator, and for nearly 50 years afterwards, the Clarendon Laboratory was a place of teaching from which research was virtually banned. Whatever may have been Professor Clifton's merits as a teacher or as a collector of the choicest scientific instruments of his day, he certainly was not a man who could have inspired young Stocker to engage in research and to take up physics as his hobby. It is perhaps not idle to speculate whether, given a more inspiring scientific environment during his earlier years, Stocker might not have become the Grand Old Man of Oxford Physics. As it is, one thinks of him as an original and therefore typical old bachelor don of the Victorian era whose sayings, doings and habits, enriched and embellished by the passage of time, will probably be handed down as a legend to posterity.

N. KURTI.

CORRIGENDUM

"An Improved Method of Numerical Ray Tracing through Electron Lenses", by G. LIEBMANN. (*This issue*, page 753.)

Equation (18e), p. 761, should read

$$Q_0 = k^2 h^2 \left\{ \left[\frac{1}{2} (h''/h) - k^2 h^2 \right] r_{n+\frac{1}{2}}^3 + (h'/h) r_{n+\frac{1}{2}}^2 r'_{n+\frac{1}{2}} - r_{n+\frac{1}{2}} r_{n+\frac{1}{2}}'^2 \right\} \Delta z.$$

instead of

$$Q_0 = k^2 h^2 \left\{ \left[\frac{1}{2} (h''/h) - \frac{1}{4} k^2 h^2 \right] r_{n+\frac{1}{2}}^3 + (h'/h) r_{n+\frac{1}{2}}^2 r'_{n+\frac{1}{2}} - r_{n+\frac{1}{2}} r_{n+\frac{1}{2}}'^2 \right\} \Delta z.$$

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